CSE 312

Foundations of Computing II

Lecture 5: Intro to Discrete Probability

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Rachel Lin, Hunter Schafer

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Music: Honne

Agenda

- Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Probability

- We want to model a non-deterministic process.
  - i.e., outcome not determined a-priori
  - E.g. throwing dice, flipping a coin…
  - We want to numerically measure likelihood of outcomes = probability.
  - We want to make complex statements about these likelihoods.
- We will not argue why a certain physical process realizes the probabilistic model we study
  - Why is the outcome of the coin flip really "random"?
- First part of class: "Discrete" probability theory
  - Experiment with finite / discrete set of outcomes.
  - Will explore countably infinite and continuous outcomes later

Sample Space

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events

$A \subseteq B$ "A is subset of B"

$x \in Y$ "x is an element of y"

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

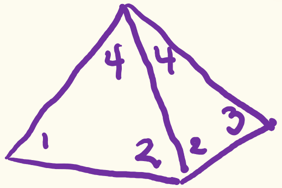
- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

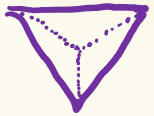
Definition. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$ (i.e., can't happen at same time)

Examples:

- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$

Example: 4-sided Dice





Suppose I roll two 4-sided dice Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. D1 = 1

B. D1 + D2 = 6

C. D1 = 2 * D2

Die 2 (D2) Ω

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice

Suppose I roll two 4-sided dice Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. D1 = 1

$$A = \{(1,1),(1,2)(1,3),(1,4)\}$$

B. D1 + D2 = 6

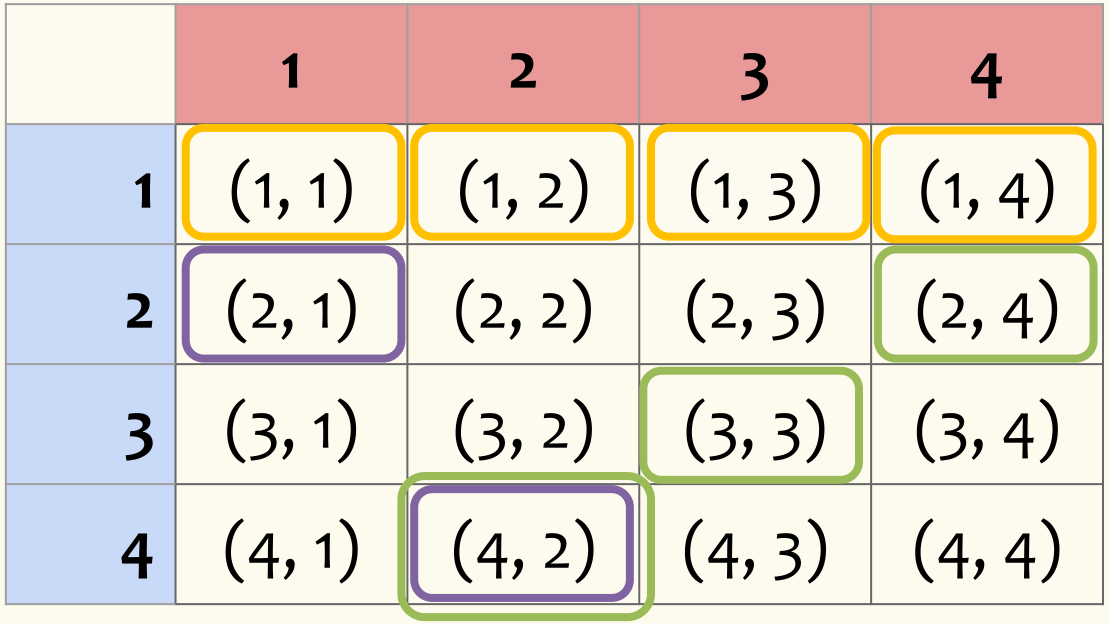
$$B = \{(2,4),(3,3)(4,2)\}$$

C. D1 = 2 * D2

$$C = \{(2,1),(4,2)\}$$

Die 2 (D2)

Die 1 (D1)



	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice, Mutual Exclusivity

N.E. $\Longleftrightarrow$ $E \cap F \neq O$

Are A and B mutually exclusive?
How about B and C?
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	A & B	B & C
(a)	Yes	Yes
✓ (b)	Yes ✓	No ✓
(c)	No	Yes
(d)	No	No

$A \cap B = \emptyset$ $\qquad B \cap C = \{(4,2)\}$

A. D1 = 1

$$A = \{(1,1),(1,2),(1,3),(1,4)\}$$

B. D1 + D2 = 6

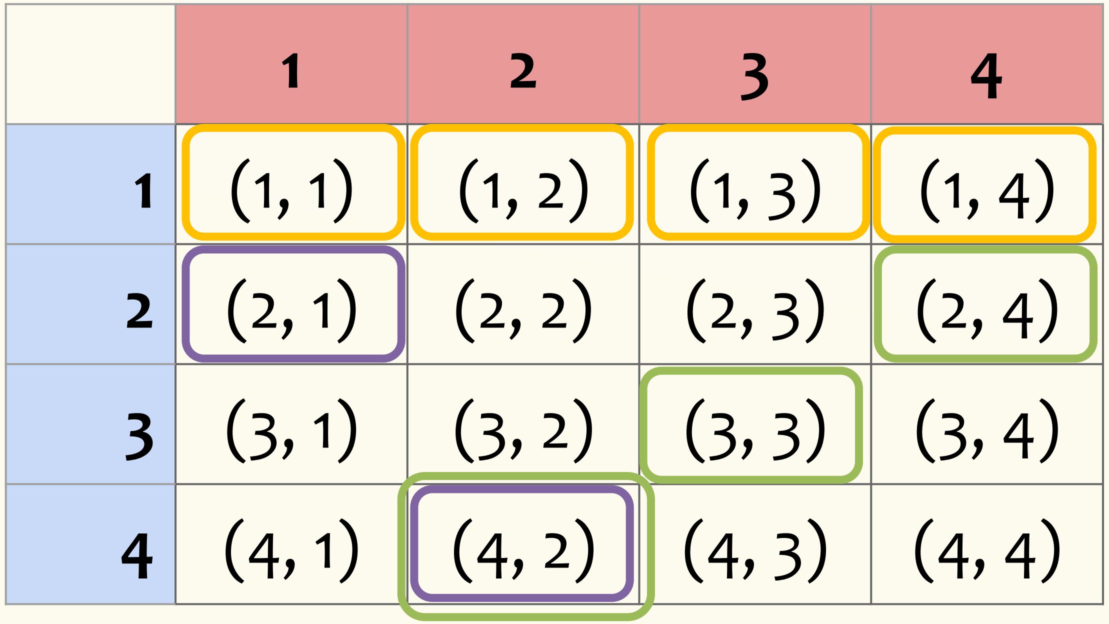
$$B = \{(2,4),(3,3),(4,2)\}$$

C. D1 = 2 * D2

$$C = \{(2,1),(4,2)\}$$

Die 2 (D2)

Die 1 (D1)



	1	**2**	**3**	**4**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
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Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function probability measure

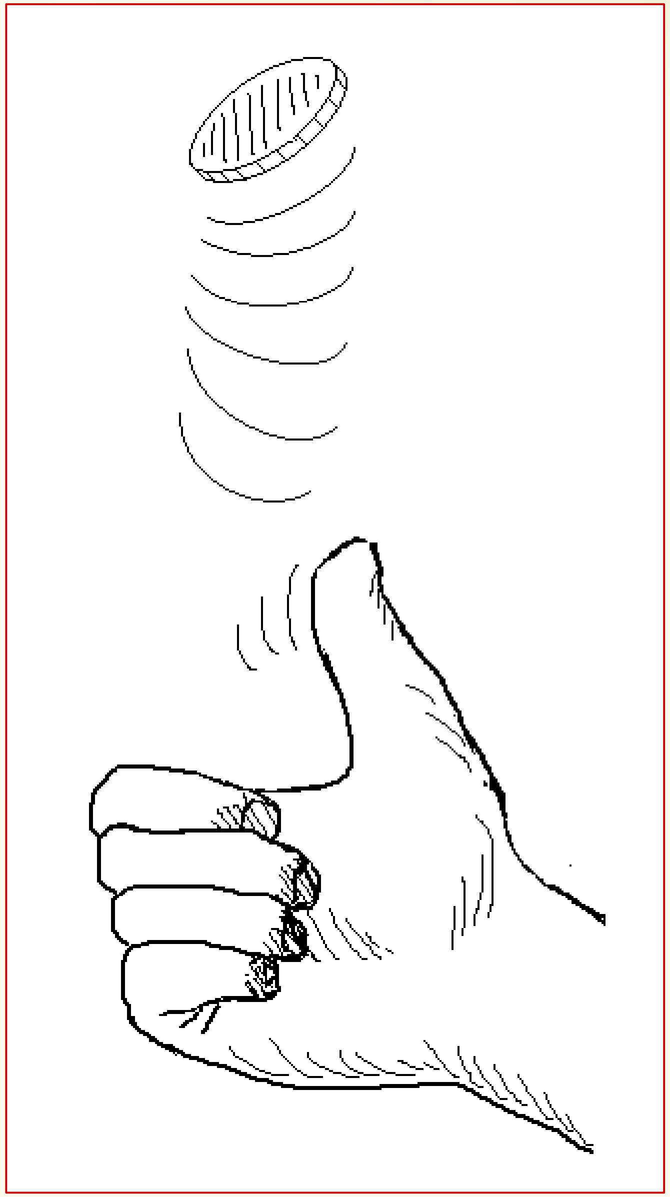
$\omega \in \Omega$

$$\mathbb{P}(\omega): \Omega \to [0, 1]$$

that maps outcomes $\omega \in \Omega$ to probabilities.

– Also use notation: $\mathbb{P}(\omega) = P(\omega) = \Pr(\omega)$

Example – Coin Tossing



Imagine we toss one coin – outcome can be **heads** or **tails.**

$$\Omega = \{\mathrm{H}, \mathrm{T}\}$$

$\mathbb{P}$? Depends! What do we want to model?!

Fair coin toss

$$\mathbb{P}(\mathrm{H}) = \mathbb{P}(\mathrm{T}) = \frac{1}{2} = 0.5$$

Example – Coin Tossing

Imagine we toss <u>one</u> coin – outcome can be **heads** or **tails.**

$$\Omega = \{\mathrm{H}, \mathrm{T}\}$$

$\mathbb{P}$? Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(\mathrm{H}) = 0.85, \qquad \mathbb{P}(\mathrm{T}) = 0.15$$

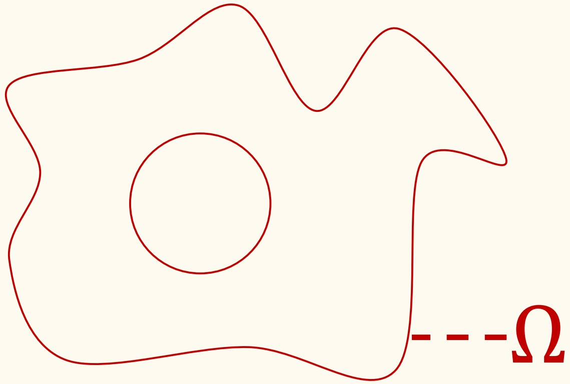
Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair $\boxed{(\Omega, \mathbb{P})}$ where:

- Ω is a set called the **sample space.**
- $\mathbb{P}$ is the **probability measure,** a function $\mathbb{P}: \Omega \to [0,1]$ such that:
  - $\mathbb{P}(x) \geq 0$ for all $x \in \Omega$
  - $\sum_{x\in\Omega} \mathbb{P}(x) = 1$

Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Uniform Probability Space

Definition. A **uniform probability space** is a pair $(\Omega, \mathbb{P})$ such that

$$\mathbb{P}(x) = \frac{1}{|\Omega|}$$

for all $x \in \Omega$.

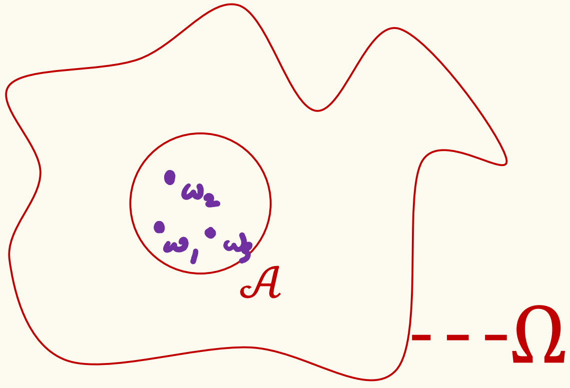
Examples:

- Fair coin $P(x) = \frac{1}{2}$
- Fair 6-sided die $P(x) = \frac{1}{6}$

Events

Definition. An **event** in a probability space $(\Omega, \mathbb{P})$ is a subset $\mathcal{A} \subseteq \Omega$. Its probability is

$$\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)$$



Convenient abuse of notation: $\mathbb{P}$ is extended to be defined over **sets.**

$$\mathbb{P}(\omega) = \mathbb{P}(\{\omega\})$$

Care if the argument is an event or outcome!

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Example: 4-sided Dice, Event Probability

$$\Pr(\omega) = \frac{1}{|\Omega|} = \frac{1}{16}$$

Think back to 4-sided die. Suppose each die is fair. What is the probability of event B? $\Pr(B) = ???$

B. D1 + D2 = 6

$B = \{(2,4), (3,3)(4,2)\}$

$$\Pr(B) = \sum_{\omega \in B} \Pr(\omega) = \sum_{\omega \in B} \frac{1}{16} = \frac{3}{16}$$

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
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4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Equally Likely Outcomes

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$, then

$$P(E) = \frac{|E|}{|\Omega|}$$

This follows from the definitions of the prob. of an event and uniform probability spaces.

$$Pr(E) = \sum_{\omega \in E} Pr(\omega) = \sum_{\omega \in E} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}$$

Example – Coin Tossing

$\Omega = \{$ sequences of 100 flips $\}$

$E = \{$ sequences w/ 50 heads $\}$

Toss a coin 100 times. Each outcome is **equally likely** (and assume the outcome of one toss does not impact another). What is the probability of seeing 50 heads?

$$\Pr(E) = \frac{|E|}{|\Omega|}$$

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(a) $\frac{1}{2}$

(b) $\frac{1}{2^{50}}$

(c) $\frac{\binom{100}{50}}{2^{100}}$

(d) Not sure

$$|\Omega| = 2^{100}$$

$$|E| = \binom{100}{50}$$

$$\Pr(E) = \frac{|E|}{|\Omega|} = \frac{\binom{100}{50}}{2^{100}}$$

Brain Break

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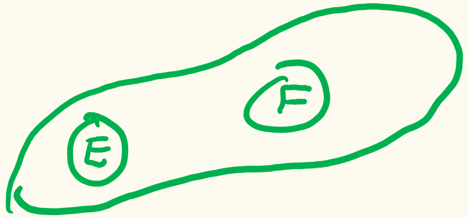
Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is more general to **any** probability space (not just uniform)

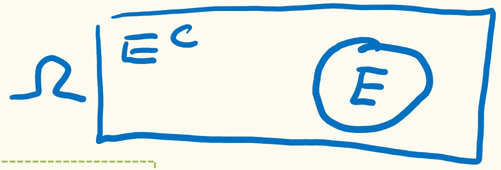
Axiom 1 (Non-negativity): $P(E) \geq 0.$

Axiom 2 (Normalization): $P(\Omega) = 1$

Axiom 3 (Countable Additivity): If *E* and *F* are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$



$$E^c = \Omega \setminus E \quad \Rightarrow \quad E \cup E^c = \Omega$$



Corollary 1 (Complementation): $P(E^c) = 1 - P(E).$

Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$

Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$$1 = \Pr(\Omega) = \Pr(E \cup E^c) = \Pr(E) + \Pr(E^c) \Rightarrow P(E^c) = 1 - P(E)$$

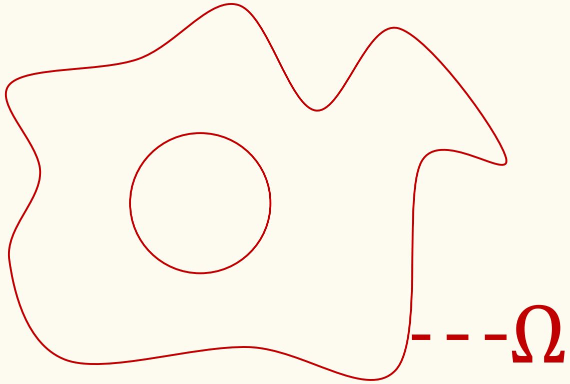
Review Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

- Ω is a set called the **sample space.**
- $\mathbb{P}$ is the **probability measure,** a function $\mathbb{P}: \Omega \to \mathbb{R}$ such that:
  - $\mathbb{P}(x) \geq 0$ for all $x \in \Omega$
  - $\sum_{x \in \Omega} \mathbb{P}(x) = 1$

Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Non-equally Likely Outcomes

Many probability spaces can have **non-equally likely outcomes.**

Examples:

- Biased Coin: $P(H) = p\ , P(T) = 1\ - P(H) = 1 - p$
- Glued coin: $P(HH) = P(TT) = 0, P(HT) = 0.5, P(TH) = 0.5$

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Example: Dice Rolls

Complementary Counting

Suppose I had a two, fair, 6-sided dice that we roll once each. What is the probability that we see *at least one 3 in the two rolls.*

$\Omega = \{\text{all pairs of dice rolls}\}$

$|\Omega| = 36$

$E = \{\text{all rolls w/ } \geq 1 \text{ three}\}$

$Pr(E) = \frac{|E|}{|\Omega|}$ ← tricky to count

$E^c = \{\text{rolls w/ no 3's}\}$

$|E^c| = 5 \cdot 5 = 25$

$$\begin{array}{cccc} 1 & & 1 & \\ 2 & & 2 & \\ 4 & & 4 & \\ 5 & & 5 & \\ 6 & & 6 & \\ \hline 5 & \cdot & 5 & = 25 \end{array}$$

$$\begin{aligned} Pr(E) &= 1 - Pr(E^c) \\ &= 1 - \frac{|E^c|}{|\Omega|} \\ &= 1 - \frac{25}{36} \\ &= \frac{11}{36} \end{aligned}$$

Example: Birthday "Paradox"

Hint: Complementary Counting

$n=23$, $\Pr(\ldots) \geq 0.5$

$n=57$, $\Pr(\ldots) \geq 0.99$

Suppose we have a collection of n people in a room. What is the probability that at least 2 people share a birthday? Assuming there are 365 possible birthdays, with uniform probability for each day.

See notes on website for explanation