Ask questions or say hi in chat before/after class

CSE 312

Foundations of Computing II

Lecture 3: Even more counting

Binomial Theorem, Inclusion-Exclusion, Pigeonhole Principle

PAUL G. ALLEN SCHOOL
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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Music: Carly Rae Jepsen

Announcement

- First Homework tonight
  - Posted by tonight on course website
  - Written and coding portion
    - Recommend typing up written solutions in LaTeX (see practice on Ed)
    - Coding solutions can be done on Ed
  - Deadline 11:59pm Next Friday
  - Submission to Gradescope. Post on EdStem if you are not enrolled in Gradescope
  - Tip: Section solutions are good examples of how to write solutions to these problems!
- Resources
  - Textbook readings can provide another perspective
  - Theorems & Definitions sheet
  - Office Hours

Agenda

- Recap & Finish Binomial Coefficients
- Binomial Theorem
- Inclusion-Exclusion
- Pigeonhole Principle

Recap of Last Time

Permutations. The number of orderings of n distinct objects

$$n! = n \times (n-1) \times \cdots \times 2 \times 1$$

Example: How many sequences in $\{1,2,3\}^3$ *with no repeating elements?*

k-Permutations. The number of orderings of **only** k out of n distinct objects

$$P(n,k)$$

$$= n \times (n-1) \times \cdots \times (n-k+1)$$

$$= \frac{n!}{(n-k)!}$$

Example: How many sequences of 5 *distinct alphabet letters from* $\{A, B, \ldots, Z\}$*?*

Combinations / Binomial Coefficient. The number of ways to select k out of n objects, where ordering of the selected k does not matter:

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

"n choose k"

Example: How many size-5 ***subsets*** *of* $\{A, B, \ldots, Z\}$*?*

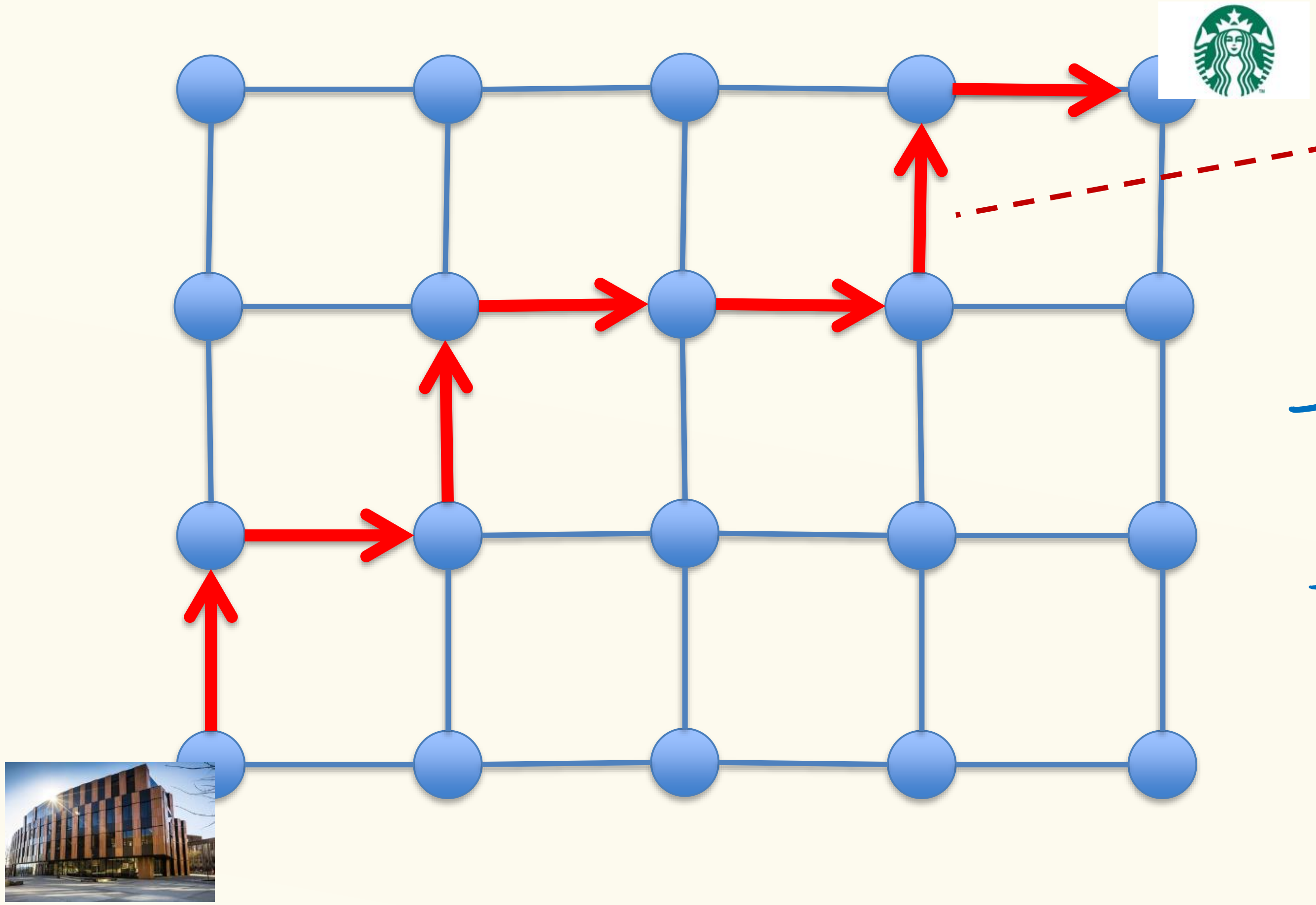
Example: How many shortest paths from Gates to Starbucks?

Example: How many solutions $(x_1, \ldots, x_k)$ *such that* $x_1, \ldots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?*

Recap* Example – Counting Paths

$$\text{Path} \in \{\uparrow, \rightarrow\}^7$$

A slightly modified example



Example path:
$(\uparrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow)$

$\_ \ \overset{\uparrow}{\_} \ \_ \ \overset{\uparrow}{\_} \ \_ \ \overset{\uparrow}{\_} \ \_$

$\{1, 2, \ldots, 7\}$

$$\binom{7}{3} = \binom{7}{4}$$

Recap Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$ Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ Pascal's Identity (This lecture)

Fact. $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ Follows from Binomial theorem (This lecture)

Recap Combinatorial vs Algebraic arguments/proofs

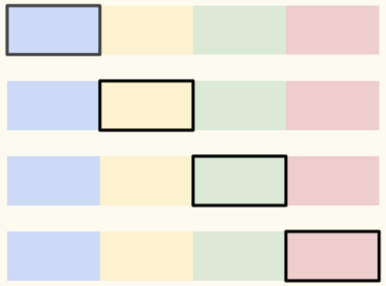
Combinatorial argument/proof

- Elegant
- Simple
- Intuitive

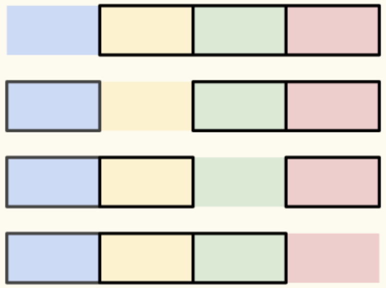
Algebraic argument/proof

- Brute force
- Less Intuitive

Argument/Proof.



$$\binom{4}{1} = 4 = \binom{4}{3}$$



Proof.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$$

Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

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Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ Pascal's Identity (Right now)

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ Follows from Binomial theorem (This lecture)

Pascal's Identities

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

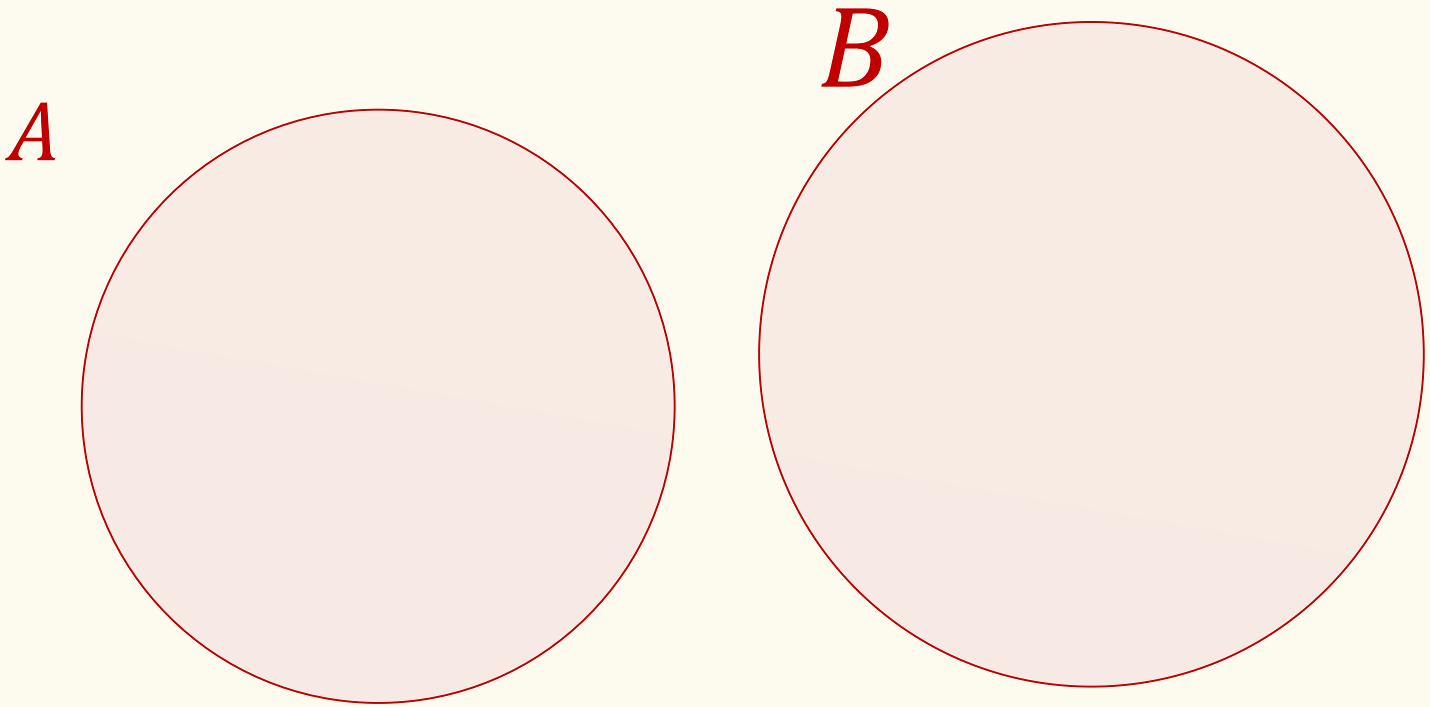
$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!} \\ &= 20\ \textit{years later} \ldots \\ &= \frac{n!}{k!\,(n-k)!} \\ &= \binom{n}{k}\end{aligned}$$

Hard work and not intuitive

Let's see a combinatorial argument

Disjoint Sets

Sets that do not contain common elements ($A \cap B = \emptyset$)



Fact. $|A \cup B| = |A| + |B|$

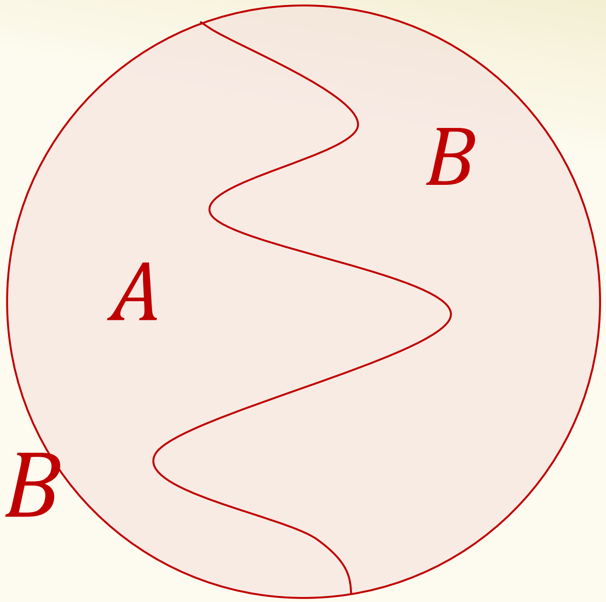
Example – Binomial Identity

$S = A \cup B$

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \quad = \quad |A| \quad + \quad |B|$

$S = A \cup B$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

e.g.: $n = 4, S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: the set of size k subsets of $[n]$ including n

$$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}$$

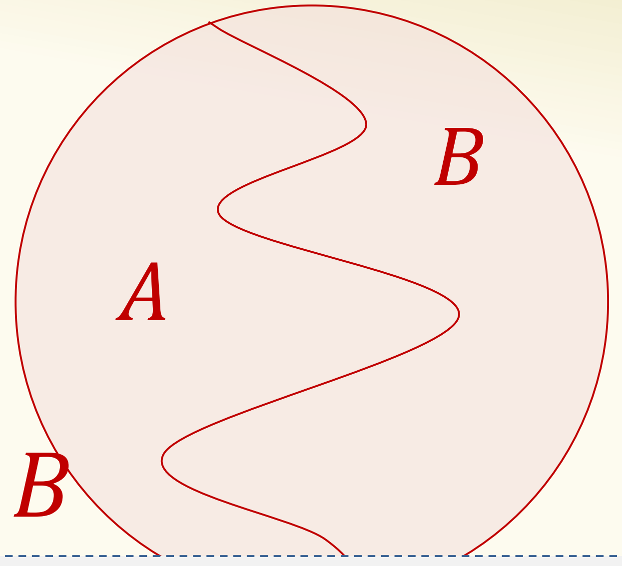
B: the set of size k subsets of $[n]$ NOT including n

$$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$$

Example – Binomial Identity

Fact. $$\underbrace{\binom{n}{k}}_{|S|} = \underbrace{\binom{n-1}{k-1}}_{|A|} + \underbrace{\binom{n-1}{k}}_{|B|}$$

$S = A \cup B$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$

A: the set of size k subsets of $[n]$ including n

n is in set, need to choose $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

B: the set of size k subsets of $[n]$ NOT including n

n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

Agenda

- Recap & Finish Binomial Coefficients
- Binomial Theorem
- Inclusion-Exclusion
- Pigeonhole Principle

Binomial Theorem: Idea

Poll: What is the coefficient for xy^3?

A. 4

B. $\binom{4}{1}$

C. $\binom{4}{3}$

D. 3

https://pollev.com/hunter312

$$(x+y)^2 = (x+y)(x+y)$$
$$= xx + xy + yx + yy$$
$$= x^2 + 2xy + y^2$$

$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$

$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

x^4 $\quad$ y^4 $\quad$ x^2y^2 $\quad$ xy^3

$\underline{x}$ _ _ _

_ $\underline{y}$ $\underline{y}$ $\underline{y}$

$\binom{4}{1} = \binom{4}{3} = 4$

Binomial Theorem: Idea

$$(x + y)^n = (x + y) \dots (x + y)$$

Each term is of the form $x^k y^{n-k}$, since each term is is made by multiplying exactly n variables, either x or y.

How many times do we get $x^k y^{n-k}$? The number of ways to choose k of the n variables we multiple to be an x (the rest will be y).

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

$n=2$ $\quad (x+y)^2 = \sum_{k=0}^{2} \binom{2}{k} x^k y^{2-k} = \binom{2}{0} x^0 y^2 + \binom{2}{1} x^1 y^1 + \binom{2}{2} x^2 y^0$

$= x^2 + 2xy + y^2$

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$\underbrace{(x+y)^n}_{} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

Brain Break

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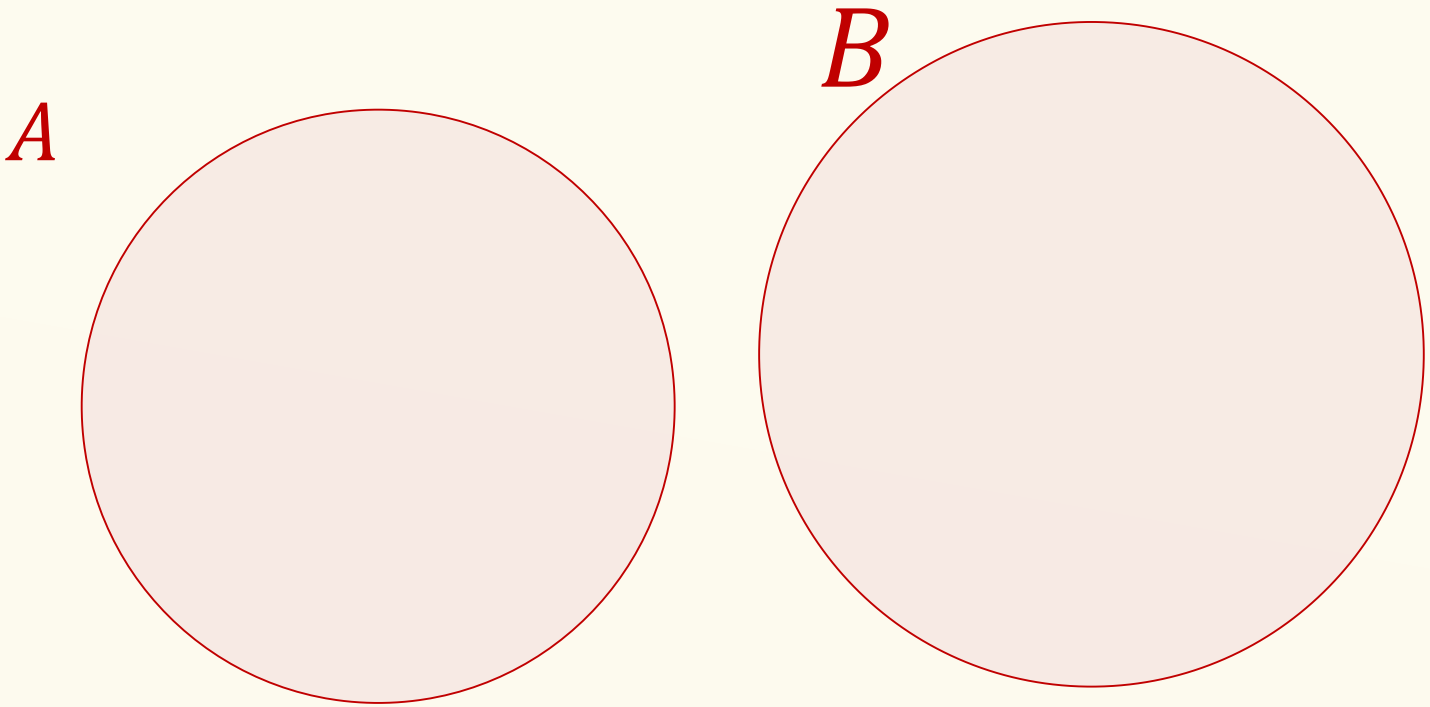
Recap Disjoint Sets

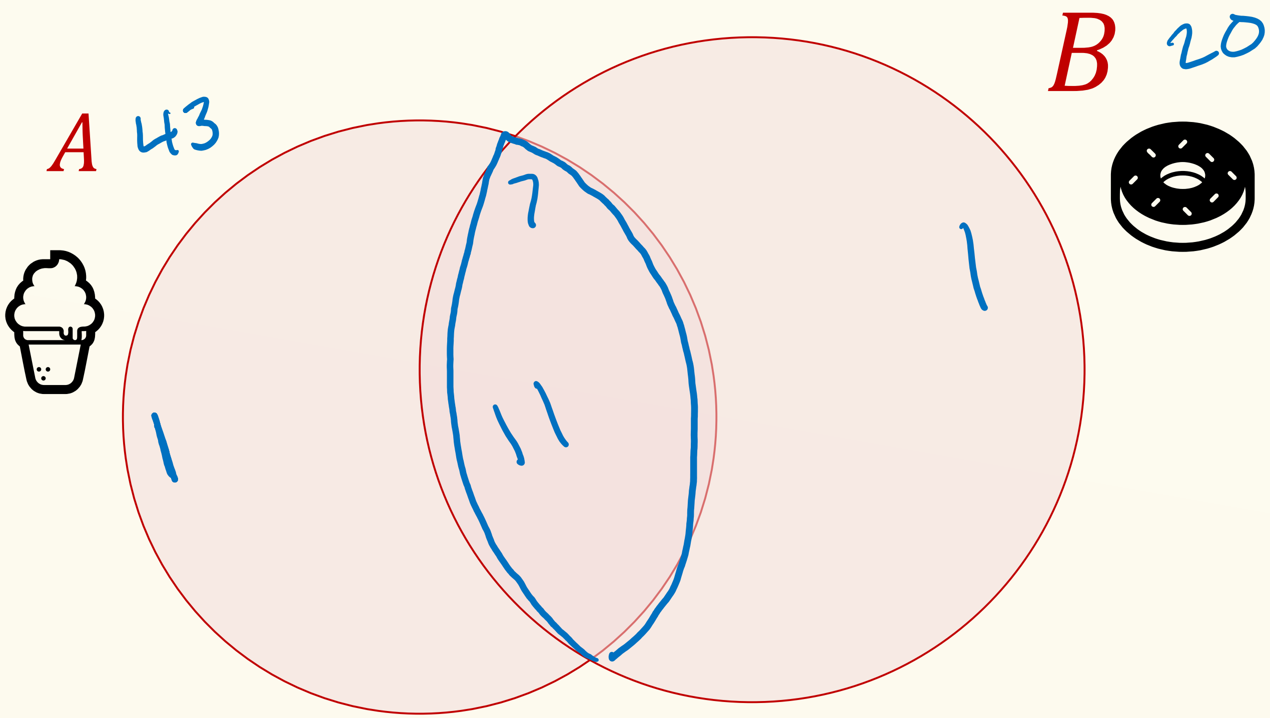
Sets that do not contain common elements ($A \cap B = \emptyset$)



Fact. $|A \cup B| = |A| + |B|$

Inclusion-Exclusion

But what if the sets are not disjoint?



$|A| = 43$
$|B| = 20$
$|A \cap B| = 7$
$|A \cup B| = ???$

$|A \cup B| = \cancel{|A| + |B| = 43 + 20 = 63}$

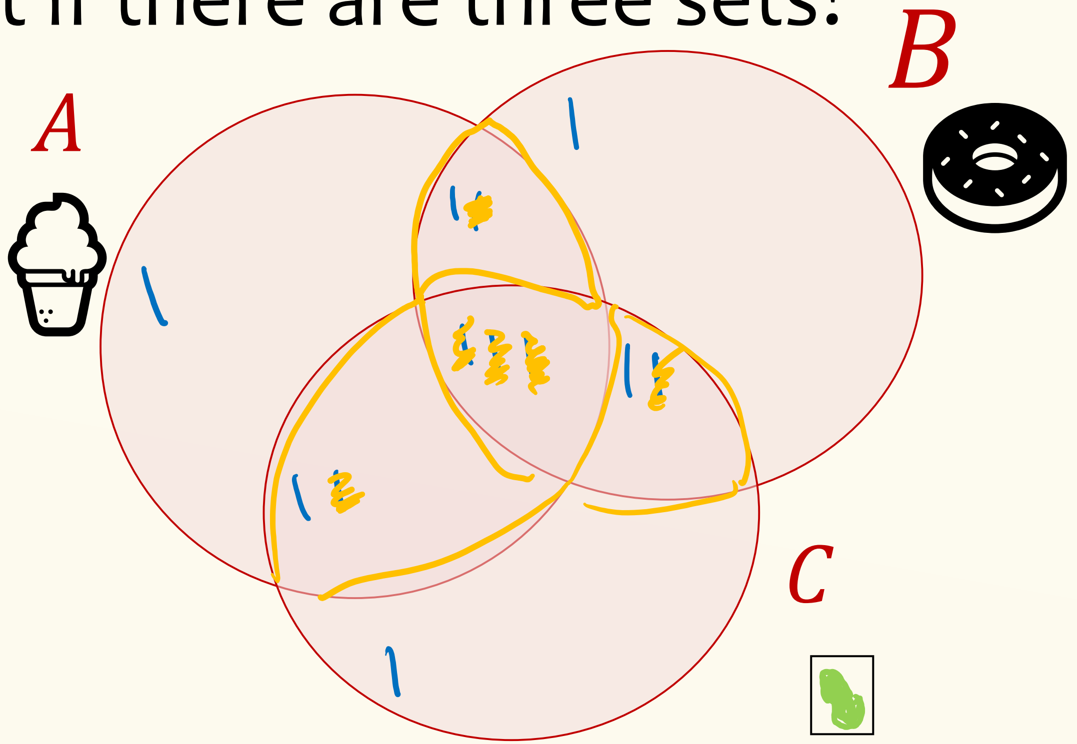
$= |A| + |B| - |A \cap B|$

Fact. $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion

Not drawn to scale

What if there are three sets?



$|A| = 43$
$|B| = 20$
$|C| = 35$
$|A \cap B| = 7$
$|A \cap C| = 16$
$|B \cap C| = 11$
$|A \cap B \cap C| = 4$
$|A \cup B \cup C| = ???$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Fact.

$$\begin{aligned} |A \cup B \cup C| = {} & |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C| \end{aligned}$$

Inclusion-Exclusion

Let A, B be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= singles - doubles + triples - quads + \ \ldots \\ &= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \ \ldots + |A_{n-1} \cap A_n|) + \ \ldots \end{aligned}$$

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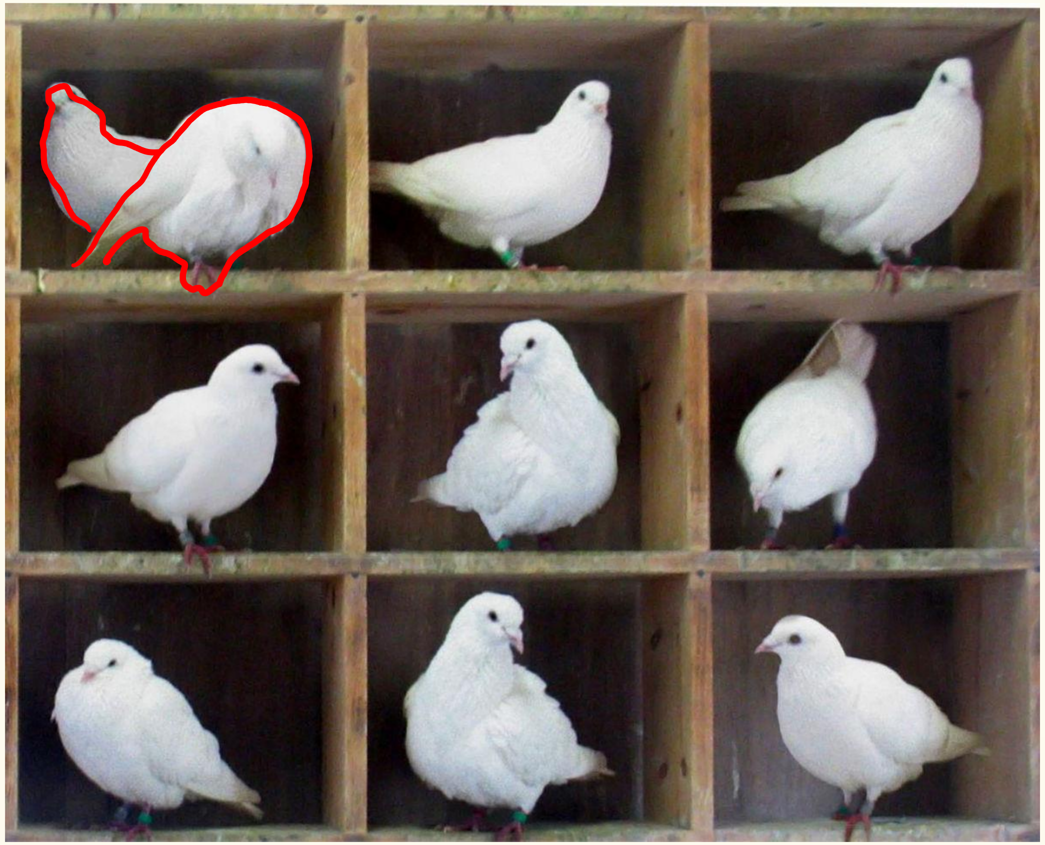
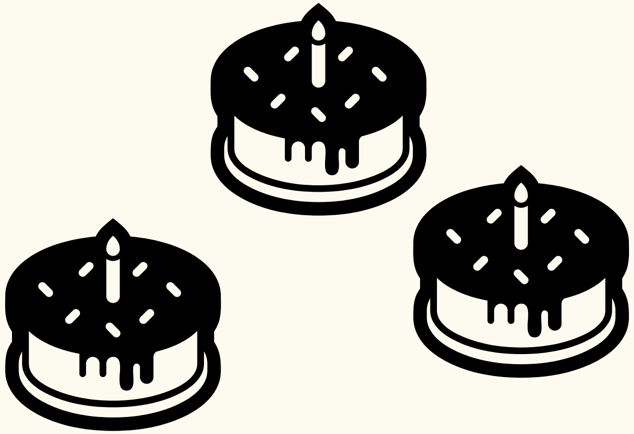
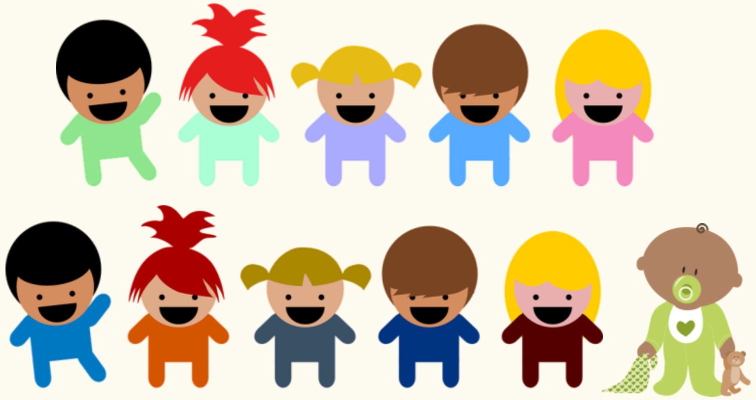
Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes



Pigeonhole Principle: Idea





If 11 children have to share 3 cakes, at least one cake must be shared by how many children?

$\frac{11}{3}$ (4 children, rounded up)

Pigeonhole Principle – More generally

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\frac{n}{k}$ pigeons!

$\exists \geq \frac{n}{k}$

Proof. Assume there are $< \frac{n}{k}$ pigeons per hole. $\perp$

Then, there are $< k\frac{n}{k} = n$ pigeons overall.

Contradiction!

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

Reason. Can't have fractional number of pigeons

Syntax reminder:

- Ceiling: $\lceil x \rceil$ is x rounded up to the nearest integer (e.g., $\lceil 2.731 \rceil = 3$)
- Floor: $\lfloor x \rfloor$ is x rounded down to the nearest integer (e.g., $\lfloor 2.731 \rfloor = 2$)

Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

$$\left\lceil \frac{367}{365} \right\rceil = \lceil 1.0001 \rceil = 2$$

Solution:

1. **367** pigeons = people
2. **365** holes = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

Pigeonhole Principle – Example (Surprising?)

7, 10523, -74, 10^{10}, ...

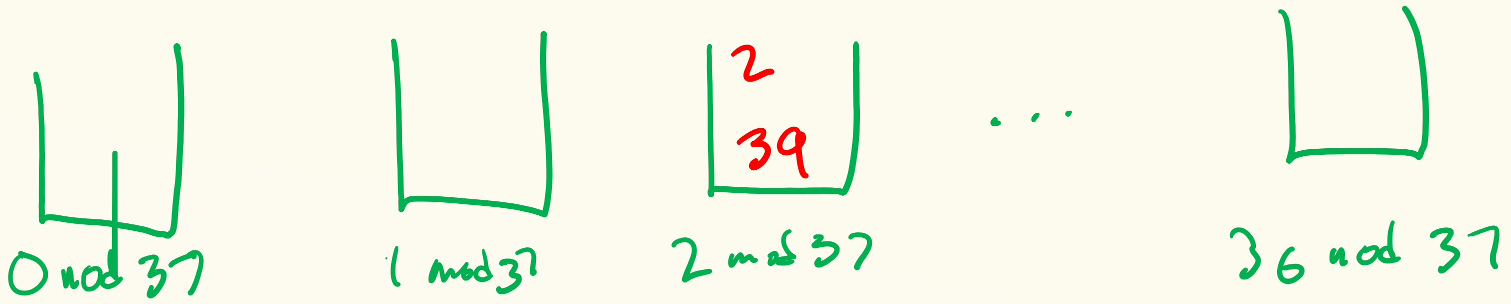
In every set S of 100 integers, there are at least ***two*** *elements whose difference is a multiple of 37.*

When solving a PHP problem:

cscript %systemroot%\system32\slmgr.vbs -skms mskms.cac.washington.edu:1688

1. Identify pigeons — 100 numbers
2. Identify pigeonholes — 37 buckets
3. Specify how pigeons are assigned to pigeonholes — $i \to i \bmod 37$
4. Apply PHP

$$\geq \left\lceil \frac{100}{37} \right\rceil = \lceil 2.5\ldots \rceil = 3$$



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