CSE 312: Foundations of Computing II

Section 8: Tail Bounds, Joint Distributions, Law of Total Expectation (and bit of conditional distributions) Solutions

1. Review of Main Concepts

(a) **Multivariate: Discrete to Continuous:**

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Independence must have	$\forall x,y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x,y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

(b) **Law of Total Probability (r.v. version):** If X is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A | X = x) p_X(x) \qquad \text{discrete } X$$

(c) **Law of Total Expectation (Event Version)**: Let X be a discrete random variable, and let events $A_1, ..., A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \, \mathbb{P}(A_i)$$

(d) **Conditional Expectation**: See table. Note that linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A]$

(e) **Law of Total Expectation (RV Version)**: Suppose X and Y are random variables. Then,

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y = y] \, p_Y(y) \qquad \text{discrete version.}$$

(f) **Conditional distributions**

	Discrete	**Continuous**
Conditional PMF/PDF	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X \mid Y = y] = \sum_x x p_{X\|Y}(x\|y)$	$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$

(g) **Continuous Law of Total Probability:**

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A | X = x) f_X(x) dx$$

(h) **Continuous Law of Total Expectation:**

$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X \mid Y = y] \, f_Y(y) dy$$

(i) **Markov's Inequality**: Let X be a non-negative random variable, and $\alpha > 0$. Then,

$$\mathbb{P}\left(X \geq \alpha\right) \leq \frac{\mathbb{E}[X]}{\alpha}$$

(j) **Chebyshev's Inequality**: Suppose Y is a random variable with $\mathbb{E}[Y] = \mu$ and Var$(Y) = \sigma^2$. Then, for any $\alpha > 0$,

$$\mathbb{P}\left(|Y - \mu| \geq \alpha\right) \leq \frac{\sigma^2}{\alpha^2}$$

(k) **Chernoff Bound (for the Binomial)**: Suppose $X \sim$ Binomial(n, p) and $\mu = np$. Then, for any $0 < \delta < 1$,

- $\mathbb{P}\left(X \geq (1+\delta)\,\mu\right) \leq e^{-\frac{\delta^2 \mu}{3}}$
- $\mathbb{P}\left(X \leq (1-\delta)\,\mu\right) \leq e^{-\frac{\delta^2 \mu}{2}}$

2. Tail bounds

Suppose $X \sim$ Binomial$(6, 0.4)$. We will bound $\mathbb{P}(X \geq 4)$ using the tail bounds we've learned, and compare this to the true result.

(a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?

Solution:

We know that the expected value of a binomial distribution is np, so: $\mathbb{P}(X \geq 4) \leq \frac{\mathbb{E}[X]}{4} = \frac{2.4}{4} = 0.6$. We can use it since X is nonnegative.

(b) Give an upper bound for this probability using Chebyshev's inequality. You may have to rearrange algebraically and it may result in a weaker bound.

Solution:

$\mathbb{P}(X \geq 4) = \mathbb{P}(X - 2.4 \geq 1.6) \leq \mathbb{P}(|X - 2.4| \geq 1.6)$ we can add those absolute value signs because that only adds more possible values, so it is an upper bound on the probability of $X - 2.4 \geq 1.6$. Then, using Chebyshev's inequality we get:
$\mathbb{P}(|X - 2.4| \geq 1.6) \leq \frac{Var(X)}{1.6^2} = \frac{1.44}{1.6^2} = 0.5625$

(c) Give an upper bound for this probability using the Chernoff bound.

Solution:

$\mathbb{P}(X \geq 4) = \mathbb{P}(X \geq (1 + \frac{2}{3})2.4) \leq e^{-(\frac{2}{3})^2 \mathbb{E}[X]/3} = e^{-4 \times 2.4/27} \approx 0.7$

(d) Give the exact probability.

Solution:

Since X is a binomial, we know it has a range from 0 to n (or in this case 0 to 6). Thus, the possible values to satisfy $X \geq 4$ are 4, 5, or 6. We plug in the PMF for each to get: $\mathbb{P}(X \geq 4) = \mathbb{P}(X = 4) + \mathbb{P}(X = 5) + \mathbb{P}(X = 6) = \binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6) + \binom{6}{6}0.4^6 \approx 0.1792$

3. Joint PMF's

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

(a) Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).

Solution:

$\Omega_X = \{0, 1\}$, $\Omega_Y = \{1, 2, 3\}$, and $\Omega_{X,Y} = \{(0, 2), (0, 3), (1, 1), (1, 3)\}$

(b) Find the marginal PMF for X, $p_X(x)$ for $x \in \Omega_X$.

Solution:

$$p_X(0) = \sum_y p_{X,Y}(0, y) = 0 + 0.2 + 0.1 = 0.3$$
$$p_X(1) = 1 - p_X(0) = 0.7$$

(c) Find the marginal PMF for Y, $p_Y(y)$ for $y \in \Omega_Y$.

Solution:

$$p_Y(1) = \sum_x p_{X,Y}(x, 1) = 0 + 0.3 = 0.3$$
$$p_Y(2) = \sum_x p_{X,Y}(x, 2) = 0.2 + 0 = 0.2$$
$$p_Y(3) = \sum_x p_{X,Y}(x, 3) = 0.1 + 0.4 = 0.5$$

(d) Are X and Y independent? Why or why not?

Solution:

No, since a necessary condition is that $\Omega_{X,Y} = \Omega_X \times \Omega_Y$.

(e) Find $\mathbb{E}\left[X^3 Y\right]$.

Solution:

Note that $X^3 = X$ since X takes values in $\{0, 1\}$.

$$\mathbb{E}\left[X^3 Y\right] = \mathbb{E}[XY] = \sum_{(x,y) \in \Omega_{X,Y}} xy p_{X,Y}(x, y) = 1 \cdot 1 \cdot 0.3 + 1 \cdot 3 \cdot 0.4 = 1.5$$

4. Trinomial Distribution

A generalization of the Binomial model is when there is a sequence of n independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let X_i be the number of times outcome i occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$.

Solution:

Same argument as for the binomial PMF:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} \prod_{i=1}^{3} p_i^{x_i} = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

where $x_1 + x_2 + x_3 = n$ and are nonnegative integers.

5. Do You "Urn" to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the i-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

(a) X_1, X_2

Solution:

Here is one way of defining the joint pmf of X_1, X_2

$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156}$$

$$\mathbb{P}(X_1 = 1, X_2 = 0) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 0 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{8}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 1) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 1 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{5}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 0) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{56}{156}$$

(b) X_1, X_2, X_3

Solution:

Instead of listing out all the individual probabilities, we could write a more compact formula for the pmf. In this problem, the denominator is always $P(13, k)$, where k is the number of random variables in the joint pmf. And the numerator is $P(5, i)$ times $P(8, j)$ where i and j are the number of 1s and 0s, respectively.

If we wish to compute $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$, then the number of 1s (i.e., white balls) is $x_1 + x_2 + x_3$, and the number of 0s (i.e., red balls) is $(1 - x_1) + (1 - x_2) + (1 - x_3)$. Then, we can write the pmf as follows:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{10!}{13!} \cdot \frac{5!}{(5 - x_1 - x_2 - x_3)!} \cdot \frac{8!}{(5 + x_1 + x_2 + x_3)!}$$

6. Successes

Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first 2 successes. Find the joint pmf of X_1 and X_2. Write an expression for $E[\sqrt{X_1X_2}]$. You can leave your answer in the form of a sum.

Solution:

X_1 and X_2 take on two particular values x_1 and x_2, when there are x_1 failures followed by one success, and then x_2 failures followed by one success. Since the Bernoulli trials are independent the joint pmf is

$$p_{X_1,X_2}(x_1,x_2) = (1-p)^{x_1}p \cdot (1-p)^{x_2}p = (1-p)^{x_1+x_2}p^2$$

for $(x_1,x_2) \in \Omega_{X_1,X_2} = \{0,1,2,\ldots\} \times \{0,1,2,\ldots\}$. By the definition of expectation

$$E[\sqrt{X_1X_2}] = \sum_{(x_1,x_2)\in\Omega_{X_1,X_2}} \sqrt{x_1x_2} \cdot (1-p)^{x_1+x_2}p^2.$$

7. Continuous joint density

The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(x+y)} & x>0, y>0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of W and V is given by

$$f_{W,V}(w,v) = \begin{cases} 2 & 0<w<v, 0<v<1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent? Are W and V independent?

Solution:

For two random variables X,Y to be independent, we must have $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x \in \Omega_X$, $y \in \Omega_Y$. Let's start with X and Y by finding their marginal PDFs. By definition, and using the fact that the joint PDF is 0 outside of $y>0$, we get:

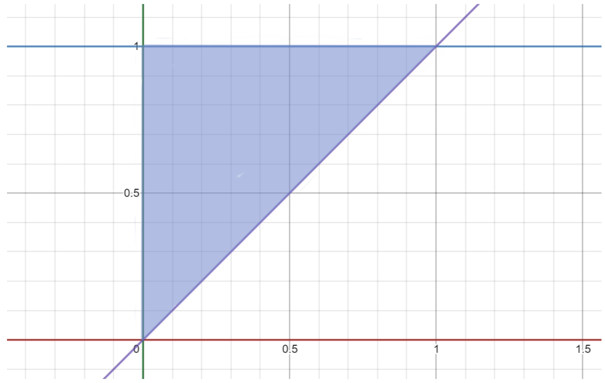
$$f_X(x) = \int_0^\infty xe^{-(x+y)}dy = e^{-x}x$$

We do the same to get the PDF of Y, again over the range $x>0$:

$$f_Y(y) = \int_0^\infty xe^{-(x+y)}dx = e^{-y}$$

Since $e^{-x}x \cdot e^{-y} = xe^{-x-y} = xe^{-(x+y)}$ for all $x,y>0$, X and Y are independent.

We can see that W and V are not independent simply by observing that $\Omega_W = (0,1)$ and $\Omega_V = (0,1)$, but $\Omega_{W,V}$ is not equal to their Cartesian product. Specifically, looking at their range of $f_{W,V}(w,v)$. Graphing it with w as the "x-axis" and v as the "y-axis", we see that :



The shaded area is where the joint pdf is strictly positive. Looking at it, we can see that it is not rectangular, and therefore it is not the case that $\Omega_{W,V} = \Omega_W \times \Omega_V$. Remember, the joint range being the Cartesian product of the marginal ranges is not sufficient for independence, but it is *necessary*. Therefore, this is enough to show that they are not independent.

8. Trapped Miner

A miner is trapped in a mine containing 3 doors.

- D_1: The 1^{st} door leads to a tunnel that will take him to safety after 3 hours.
- D_2: The 2^{nd} door leads to a tunnel that returns him to the mine after 5 hours.
- D_3: The 3^{rd} door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters $(12, \frac{1}{3})$.

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

Solution:

Let T = number of hours for the miner to reach safety. (T is a random variable)
Let D_i be the event the i^{th} door is chosen. $i \in \{1, 2, 3\}$. Finally, let T_3 be the time it takes to return to the mine in the third case only (a random variable). Note that the expectation of T_3 is $12 * \frac{1}{3}$ because it is binomially distributed with parameters $n = 12, p = 13$. By Law of Total Expectation, linearity of expectation, and by applying the conditional expectations given by the problem statement:

$$\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[T \mid D_1]\,\mathbb{P}(D_1) + \mathbb{E}[T \mid D_2]\,\mathbb{P}(D_2) + \mathbb{E}[T \mid D_3]\,\mathbb{P}(D_3) \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3 + T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3] + \mathbb{E}[T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (4 + \mathbb{E}[T]) \cdot \frac{1}{3}
\end{aligned}$$

Solving this equation for $\mathbb{E}[T]$, we get

$$\mathbb{E}[T] = 12$$

Therefore, the expected number of hours for this miner to reach safety is 12.

9. Lemonade Stand

Suppose I run a lemonade stand, which costs me \$100 a day to operate. I sell a drink of lemonade for \$20. Every person who walks by my stand either buys a drink or doesn't (no one buys more than one). If it is raining, n_1 people walk by my stand, and each buys a drink independently with probability p_1. If it isn't raining, n_2 people walk by my stand, and each buys a drink independently with probability p_2. It rains each day with probability p_3, independently of every other day. Let X be my profit over the next week. In terms of n_1, n_2, p_1, p_2 and p_3, what is $\mathbb{E}[X]$?

Solution:

Let R be the event it rains. Let X_i be how many drinks I sell on day i for $i = 1, ..., 7$. We are interested in $X = \sum_{i=1}^{7} (20X_i - 100)$. We have $X_i|R \sim$ Binomial(n_1, p_1), so $\mathbb{E}[X_i \mid R] = n_1 p_1$. Similarly, $X_i|R^C \sim$ Binomial(n_2, p_2), so $\mathbb{E}\left[X_i \mid R^C\right] = n_2 p_2$. By the law of total expectation,

$$\mu = \mathbb{E}[X_i] = \mathbb{E}[X_i \mid R] \, \mathbb{P}(R) + \mathbb{E}\left[X_i \mid R^C\right] \mathbb{P}(R^C) = n_1 p_1 p_3 + n_2 p_2 (1 - p_3)$$

Hence, by linearity of expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{7} (20X_i - 100)\right] = 20 \sum_{i=1}^{7} \mathbb{E}[X_i] - 700 = 140\mu - 700$$
$$= 140 \cdot (n_1 p_1 p_3 + n_2 p_2 (1 - p_3)) - 700.$$

10. 3 points on a line

(This problem uses the continuous law of total probability which has not yet be covered in class.) Three points X_1, X_2, X_3 are selected at random on a line L (continuous independent uniform distributions). What is the probability that X_2 lies between X_1 and X_3?

Solution:

Let $X_1, X_2, X_3 \sim Unif(0, 1)$.

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < X_2 < X_3 \mid X_2 = x) \, f_{X_2}(x) \, dx && \text{Continuous LoTP} \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x, X_3 > x) \, f_{X_2}(x) \, dx && \text{Independence of } X_1, X_2, X_3 \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x) \, \mathbb{P}(x < X_3) \, f_{X_2}(x) \, dx && \text{Independence of } X_1, X_3 \\
&= \int_{-\infty}^{\infty} F_{X_1}(x) \, (1 - F_{X_3}(x)) \, f_{X_2}(x) \, dx \\
&= \int_{0}^{1} x \, (1 - x) \, 1 \, dx \\
&= \frac{x^2}{2} - \frac{x^3}{3} \bigg|_0^1 = \frac{1}{6}
\end{aligned}$$