CSE 312: Foundations of Computing II

Section 8: Tail Bounds, Joint Distributions, Law of Total Expectation (and bit of conditional distributions)

1. Review of Main Concepts

(a) **Multivariate: Discrete to Continuous:**

<table>
<thead>
<tr>
<th></th>
<th>Discrete</th>
<th>Continuous</th>
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<tbody>
<tr>
<td><strong>Joint PMF/PDF</strong></td>
<td>( p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y) )</td>
<td>( f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y) )</td>
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<tr>
<td><strong>Joint range/support</strong></td>
<td>( {(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) &gt; 0} )</td>
<td>( {(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) &gt; 0} )</td>
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<tr>
<td><strong>Joint CDF</strong></td>
<td>( F_{X,Y}(x,y) = \sum_{t&lt;s} p_{X,Y}(t,s) )</td>
<td>( F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(t,s) , ds , dt )</td>
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<tr>
<td><strong>Normalization</strong></td>
<td>( \sum_{x,y} p_{X,Y}(x,y) = 1 )</td>
<td>( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) , dx , dy = 1 )</td>
</tr>
<tr>
<td><strong>Marginal PMF/PDF</strong></td>
<td>( p_X(x) = \sum_y p_{X,Y}(x,y) )</td>
<td>( f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) , dy )</td>
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<tr>
<td><strong>Expectation</strong></td>
<td>( \mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y)p_{X,Y}(x,y) )</td>
<td>( \mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) , dx , dy )</td>
</tr>
<tr>
<td><strong>Independence</strong></td>
<td>( \forall x,y, p_{X,Y}(x,y) = p_X(x)p_Y(y) )</td>
<td>( \forall x,y, f_{X,Y}(x,y) = f_X(x)f_Y(y) )</td>
</tr>
<tr>
<td>&amp; ( \Omega_{X,Y} = \Omega_X \times \Omega_Y )</td>
<td>( \Omega_{X,Y} = \Omega_X \times \Omega_Y )</td>
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</table>

(b) **Law of Total Probability (r.v. version):** If \( X \) is a discrete random variable, then

\[
\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A \mid X = x)p_X(x)
\]

discrete \( X \)

(c) **Law of Total Expectation (Event Version):** Let \( X \) be a discrete random variable, and let events \( A_1, \ldots, A_n \) partition the sample space. Then,

\[
\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \mathbb{P}(A_i)
\]

(d) **Conditional Expectation:** See table. Note that linearity of expectation still applies to conditional expectation: \( \mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A] \)

(e) **Law of Total Expectation (RV Version):** Suppose \( X \) and \( Y \) are random variables. Then,

\[
\mathbb{E}[X] = \sum_{y} \mathbb{E}[X \mid Y = y]p_Y(y)
\]
discrete version.

(f) **Conditional distributions**

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<thead>
<tr>
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<th>Discrete</th>
<th>Continuous</th>
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<tr>
<td><strong>Conditional PMF/PDF</strong></td>
<td>( p_{X \mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} )</td>
<td>( f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} )</td>
</tr>
<tr>
<td><strong>Conditional Expectation</strong></td>
<td>( \mathbb{E}[X \mid Y = y] = \sum_x xp_{X \mid Y}(x \mid y) )</td>
<td>( \mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y) , dx )</td>
</tr>
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</table>

(g) **Continuous Law of Total Probability:**

\[
\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A \mid X = x)f_X(x) \, dx
\]

(h) **Continuous Law of Total Expectation:**

\[
\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X \mid Y = y]f_Y(y) \, dy
\]
(i) **Markov’s Inequality**: Let $X$ be a non-negative random variable, and $\alpha > 0$. Then,

$$
P(X \geq \alpha) \leq \frac{E[X]}{\alpha}
$$

(j) **Chebyshev’s Inequality**: Suppose $Y$ is a random variable with $E[Y] = \mu$ and $\text{Var}(Y) = \sigma^2$. Then, for any $\alpha > 0$,

$$
P(|Y - \mu| \geq \alpha) \leq \frac{\sigma^2}{\alpha^2}
$$

(k) **Chernoff Bound (for the Binomial)**: Suppose $X \sim \text{Binomial}(n, p)$ and $\mu = np$. Then, for any $0 < \delta < 1$,

- $P(X \geq (1 + \delta) \mu) \leq e^{-\frac{\delta^2 \mu}{3}}$
- $P(X \leq (1 - \delta) \mu) \leq e^{-\frac{\delta^2 \mu}{2}}$

2. **Tail bounds**
Suppose $X \sim \text{Binomial}(6, 0.4)$. We will bound $P(X \geq 4)$ using the tail bounds we’ve learned, and compare this to the true result.

(a) Give an upper bound for this probability using Markov’s inequality. Why can we use Markov’s inequality?

(b) Give an upper bound for this probability using Chebyshev’s inequality. You may have to rearrange algebraically and it may result in a weaker bound.

(c) Give an upper bound for this probability using the Chernoff bound.

(d) Give the exact probability.

3. **Joint PMF’s**
Suppose $X$ and $Y$ have the following joint PMF:

<table>
<thead>
<tr>
<th>$X/Y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>0</td>
<td>0.4</td>
</tr>
</tbody>
</table>

(a) Identify the range of $X$ ($\Omega_X$), the range of $Y$ ($\Omega_Y$), and their joint range ($\Omega_{X,Y}$).

(b) Find the marginal PMF for $X$, $p_X(x)$ for $x \in \Omega_X$.

(c) Find the marginal PMF for $Y$, $p_Y(y)$ for $y \in \Omega_Y$.

(d) Are $X$ and $Y$ independent? Why or why not?

(e) Find $E[X^3Y]$.
4. Trinomial Distribution
A generalization of the Binomial model is when there is a sequence of \( n \) independent trials, but with three outcomes, where \( \mathbb{P}(\text{outcome } i) = p_i \) for \( i = 1, 2, 3 \) and of course \( p_1 + p_2 + p_3 = 1 \). Let \( X_i \) be the number of times outcome \( i \) occurred for \( i = 1, 2, 3 \), where \( X_1 + X_2 + X_3 = n \). Find the joint PMF \( p_{X_1,X_2,X_3}(x_1, x_2, x_3) \) and specify its value for all \( x_1, x_2, x_3 \in \mathbb{R} \).

5. Do You “Urn” to Learn More About Probability?
Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let \( X_i = 1 \) if the \( i \)-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of
(a) \( X_1, X_2 \)
(b) \( X_1, X_2, X_3 \)

6. Successes
Consider a sequence of independent Bernoulli trials, each of which is a success with probability \( p \). Let \( X_1 \) be the number of failures preceding the first success, and let \( X_2 \) be the number of failures between the first 2 successes. Find the joint pmf of \( X_1 \) and \( X_2 \). Write an expression for \( E[\sqrt{X_1X_2}] \). You can leave your answer in the form of a sum.

7. Continuous joint density
The joint density of \( X \) and \( Y \) is given by
\[
f_{X,Y}(x, y) = \begin{cases} 
    xe^{-(x+y)} & x > 0, y > 0 \\
    0 & \text{otherwise}.
\end{cases}
\]
and the joint density of \( W \) and \( V \) is given by
\[
f_{W,V}(w, v) = \begin{cases} 
    2 & 0 < w < v, 0 < v < 1 \\
    0 & \text{otherwise}.
\end{cases}
\]
Are \( X \) and \( Y \) independent? Are \( W \) and \( V \) independent?

8. Trapped Miner
A miner is trapped in a mine containing 3 doors.
- \( D_1 \): The 1\textsuperscript{st} door leads to a tunnel that will take him to safety after 3 hours.
- \( D_2 \): The 2\textsuperscript{nd} door leads to a tunnel that returns him to the mine after 5 hours.
- \( D_3 \): The 3\textsuperscript{rd} door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters \( (12, \frac{1}{3}) \).

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

9. Lemonade Stand
Suppose I run a lemonade stand, which costs me $100 a day to operate. I sell a drink of lemonade for $20. Every person who walks by my stand either buys a drink or doesn’t (no one buys more than one). If it is raining, \( n_1 \) people walk by my stand, and each buys a drink independently with probability \( p_1 \). If it isn’t raining, \( n_2 \) people
walk by my stand, and each buys a drink independently with probability $p_2$. It rains each day with probability $p_3$, independently of every other day. Let $X$ be my profit over the next week. In terms of $n_1, n_2, p_1, p_2$ and $p_3$, what is $E[X]$?

10. 3 points on a line
(This problem uses the continuous law of total probability which has not yet be covered in class.) Three points $X_1, X_2, X_3$ are selected at random on a line $L$ (continuous independent uniform distributions). What is the probability that $X_2$ lies between $X_1$ and $X_3$?