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Joint Distributions

CSE 312 Summer 21
Lecture 18

Announcements

Real World 1 grades have been released.

Real World 2 due next Wednesday.

Review Summary 3 due next Friday.

Details on the Final to be announced soon.

Today

A somewhat out-of-place lecture.

When we introduced multiple random variables, we've always had them be independent.

Because it's hard to deal with non-independent random variables.

Today is a crash-course in the toolkit for when you have multiple random variables, and they aren't independent.

Going to focus on discrete RVs.

Joint PMF, support

For two (discrete) random variables X, Y their joint pmf

$$\boldsymbol{p_{X,Y}(x,y) = \mathbb{P}(X = x \cap Y = y)}$$

When X, Y are independent then $p_{X,Y}(x,y) = p_X(x)p_Y(y)$.

For two (discrete) random variables X, Y their joint support

$$\boldsymbol{\Omega(X,Y) = \{(a,b): p_{X,Y}(a,b) > 0\} \subseteq \Omega(X) \times \Omega(Y)}$$

Examples

Roll a blue die and a red die. Each die is 4-sided. Let X be the blue die's result and Y be the red die's result.
Each die (individually) is fair. But not all results are equally likely when looking at them both together.

$p_{X,Y}$	$X=1$	$X=2$	$X=3$	$X=4$	
$Y=1$	1/16	1/16	1/16	1/16	1/4
$Y=2$	3/16	0	0	1/16	1/4
$Y=3$	0	2/16	0	2/16	1/4
$Y=4$	0	1/16	3/16	0	1/4
	1/4	1/4	1/4	1/4	

$$p_{X,Y}(1,2) = \frac{3}{16}$$

$p_{X,Y}(4,4) = 0$

$p_X(1) = \mathbb{P}(X=1) = \mathbb{P}(X=1 \mid Y=1)\cdot\mathbb{P}(Y=1)$
$+ \mathbb{P}(X=1 \mid Y=2)\cdot\mathbb{P}(Y=2)$
$+ \mathbb{P}(X=1 \mid Y=3)\cdot\mathbb{P}(Y=3)$
$+ \mathbb{P}(X=1 \mid Y=4)\cdot\mathbb{P}(Y=4)$

$p_X(1) = \frac{4}{16} = \frac{1}{4}$

Marginals

What if I just want to talk about X?

Well, use the law of total probability:

$\mathbb{P}(X = k) = \sum_{\text{partition } \{E_i\}} \mathbb{P}(X = k | E_i)\mathbb{P}(E_i)$

and use E_i to be possible outcomes for Y for the dice example

$\mathbb{P}(X = k) = \sum_{\ell=1}^{4} \mathbb{P}(X = k \,| Y = \ell)\mathbb{P}(Y = \ell)$

$= \sum_{\ell=1}^{4} \mathbb{P}(X = k \cap Y = \ell)$

$p_X(k) = \sum_{\ell=1}^{4} p_{X,Y}(k, \ell)$

$p_X(k)$ is called the "marginal" distribution for X (because we "marginalized" Y)
it's the same pmf we've always used; the name emphasizes we have gotten rid of one of the variables.

Marginals

$p_X(k) = \sum_{\ell=1}^{4} p_{XY}(k,\ell)$

So

$$p_X(2) = \frac{1}{16} + 0 + \frac{2}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4}$$

$p_{X,Y}$	X=1	X=2	X=3	X=4
Y=1	1/16	1/16	1/16	1/16
Y=2	3/16	0	0	1/16
Y=3	0	2/16	0	2/16
Y=4	0	1/16	3/16	0

Different dice

Roll two fair dice independently. Let U be the minimum of the two rolls and V be the maximum

Are U and V independent? No

Write the joint distribution in the table

What's $p_U(z)$? (the marginal for U)

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1	1/16	0	0	0
V=2	2/16	1/16	0	0
V=3	2/16	2/16	1/16	0
V=4	2/16	2/16	2/16	1/16
$p_U(1)$	7/16	5/16	3/16	1/16

Fill out the poll everywhere so Kushal knows how long to explain
Go to pollev.com/cse312su21

Different dice

Roll two fair dice independently. Let U be the minimum of the two rolls and V be the maximum

$$p_U(z) = \begin{cases} \frac{7}{16} & \text{if } z = 1 \\ \frac{5}{16} & \text{if } z = 2 \\ \frac{3}{16} & \text{if } z = 3 \\ \frac{1}{16} & \text{if } z = 4 \\ 0 & \text{otherwise} \end{cases}$$

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1	1/16	0	0	0
V=2	2/16	1/16	0	0
V=3	2/16	2/16	1/16	0
V=4	2/16	2/16	2/16	1/16

$$\sum_{(s,t) \in \Omega(X,Y)} p_{X,Y}(s,t) = 1$$

Joint Expectation

Expectations of joint functions

For a function $g(X, Y)$, the expectation can be written in terms of the joint pmf.

$$\mathbb{E}[g(X, Y)] = \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} g(x, y) \cdot p_{X,Y}(x, y)$$

This definition hopefully isn't surprising at this point (it's the value of g times the probability g takes on that value), but it's good to review it.

$$E[XY] = \sum_{x} \sum_{y} x \cdot y \cdot p_{x,y}(x, y)$$

Conditional Expectations

Waaaaaay back when, we said conditioning on an event creates a new probability space, with all the laws holding.

So, we can define things like "conditional expectations" which is the expectation of a random variable in that new probability space.

r.v Event

$$\mathbb{E}[X|E] = \sum_{x \in \Omega} x \cdot \mathbb{P}(X = x|E)$$

$$\mathbb{E}[X|Y = y] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X = x|Y = y)$$

Conditional Expectations

All your favorite theorems are still true.

For example, linearity of expectation still holds

$$\mathbb{E}[(aX + bY + c)\ |E] = a\mathbb{E}[X|E] + b\mathbb{E}[Y|E] + c$$

event

Law of Total Expectation

$P(A) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)$

Sample space was partitioned as $B_1, B_2 \ldots B_n$

Let $A_1, A_2, \ldots, A_k$ be a partition of the sample space, then

$$\mathbb{E}[X] = \sum_{i=1}^{k} \mathbb{E}[X|A_i]\mathbb{P}(A_i)$$

Let X, Y be discrete random variables, then

$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X|Y = y]\mathbb{P}(Y = y)$$

Similar in form to law of total probability, and the proof goes that way as well.

LTE

You will flip 2 (independent, fair coins). Call the number of heads $\boxed{X}$. Then (independently of the coin flips) draw a ~~geometric~~ exponential random variable Y from the distribution $\cancel{\text{Geo}}$ Exp$(\underline{X+1})$.

What is $\mathbb{E}[Y]$?

$\mathbb{P}(X=0) = 1/4$ then $Y \sim \text{Geo}(0+1)$ $\quad \mathbb{E}[Y] = \frac{1}{1}$

$\mathbb{P}(X=1) = 1/2$ then $Y \sim \text{Geo}(1+1)$ $\quad \mathbb{E}[Y] = \frac{1}{2}$

$\mathbb{P}(X=2) = 1/4$ then $Y \sim \text{Geo}(2+1)$ $\quad \mathbb{E}[Y] = \frac{1}{3}$

LTE

Exponential

You will flip 2 (independent, fair coins). Call the number of heads X. Then (independently of the coin flips) draw a ~~geometric~~ random variable Y from the distribution $\cancel{\text{Geo}}(X+1)$.

Exp $(X+1)$

$\cancel{E[Y] = \frac{1}{X+1}}$

What is $\mathbb{E}[Y]$?

$$\mathbb{E}[Y]$$

$$= \mathbb{E}[Y|X=0]\mathbb{P}(X=0) + \mathbb{E}[Y|X=1]\mathbb{P}(X=1) + \mathbb{E}[Y|X=2]\mathbb{P}(X=2)$$

$$= \mathbb{E}[Y|X=0] \cdot \frac{1}{4} + \mathbb{E}[Y|X=1] \cdot \frac{1}{2} + \mathbb{E}[Y|X=2] \cdot \frac{1}{4}$$

$$= \frac{1}{0+1} \cdot \frac{1}{4} + \frac{1}{1+1} \cdot \frac{1}{2} + \frac{1}{2+1} \cdot \frac{1}{4} = \frac{7}{12}.$$

Analogues for continuous

Everything we saw today has a continuous version.

There are "no surprises"– replace pmf with pdf and sums with integrals.

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = P(X=x, Y=y)$	$f_{X,Y}(x,y) \neq P(X=x, Y=y)$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x} \sum_{s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\,ds\,dt$
Normalization	$\sum_{x} \sum_{y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_{y} p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$
Expectation	$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$
Conditional PMF/PDF	$p_{X \mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y=y] = \sum_{x} x\, p_{X \mid Y}(x \mid y)$	$E[X \mid Y=y] = \int_{-\infty}^{\infty} x\, f_{X \mid Y}(x \mid y)\,dx$
Independence	$\forall x, y,\ p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y,\ f_{X,Y}(x,y) = f_X(x) f_Y(y)$

Covariance

We sometimes want to measure how "intertwined" X and Y are – how much knowing about one of them will affect the other.

If X turns out "big" how likely is it that Y will be "big" how much do they "vary together"

Covariance

$$\mathbf{Cov(X, Y)} = \mathbb{E}[(\underline{X - \mathbb{E}[X]})(\underline{Y - \mathbb{E}[Y]})] = \mathbb{E}[\underline{XY}] - \mathbb{E}[X]\mathbb{E}[Y]$$

$E[XY] = E[X] \cdot E[Y]$ for independent r.v.

Covariance

$Var(X+Y) = Var(X) + Var(Y)$
if X, Y are independent

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$$

That's consistent with our previous knowledge for independent variables. (for X, Y independent, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$).

You and your friend are playing a game, you flip a coin: if heads you pay your friend a dollar, if tails they pay you a dollar. Let X be your profit and Y be your friend's profit.

What is $\text{Var}(X+Y)$?

$E[X] = 0 = \frac{1}{2}(-1) + \frac{1}{2}(1)$

$E[X+Y] = E[X] + E[Y] = 0 + 0 = 0$

Covariance

You and your friend are playing a game, you flip a coin: if heads you pay your friend a dollar, if tails they pay you a dollar. Let X be your profit and Y be your friend's profit.

What is $\text{Var}(X+Y)$?

$\text{Var}(X) = \text{Var}(Y) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1 - 0^2 = 1$

$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$\mathbb{E}[XY] = \frac{1}{2} \cdot (-1 \cdot 1) + \frac{1}{2}(1 \cdot -1) = -1$

$\text{Cov}(X,Y) = -1 - 0 \cdot 0 = -1.$

$\text{Var}(X+Y) = 1 + 1 + 2 \cdot -1 = 0$