Random Variables

CSE 312 Summer 21
Lecture 8

Conditional Independence

Conditional Independence

We say A and B are conditionally independent on C if

$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$

i.e. if you condition on C, they are independent.

Conditional Independence

Two events A, B are independent conditioned on C if $\mathbb{P}(C) \neq 0$ and

$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C) \cdot \mathbb{P}(B \mid C)$$

Conditional Independence Example

You have two coins. Coin A is fair, coin B comes up heads with probability 0.85.

You will roll a (fair) die, if the result is odd flip coin A twice (independently); if the result is even flip coin B twice (independently)

Let C_1 be the event "the first flip is heads", C_2 be the event "the second flip is heads", O be the event "the die was odd"

Are C_1 and C_2 independent? Are they independent conditioned on O?

(Unconditioned) Independence

$\mathbb{P}(C_1) = \mathbb{P}(O)\mathbb{P}(C_1|O) + \mathbb{P}(\bar{O})\mathbb{P}(C_1|\bar{O})$

$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0.85 = .675$

$\mathbb{P}(C_2) = .675$ (the same formula works)

$\mathbb{P}(C_1)\mathbb{P}(C_2) = .675^2 = .455625$

$\mathbb{P}(C_1 \cap C_2) = \mathbb{P}(O)\mathbb{P}(C_1 \cap C_2|O) + \mathbb{P}(\bar{O})\mathbb{P}(C_1 \cap C_2|\bar{O})$

$= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot .85^2 = .48625$

Those aren't the same! They're not independent!

Intuition: seeing a head gives you information – information that it's more likely you got the biased coin and so the next head is more likely.

Conditional Independence

$\mathbb{P}(C_1|O) = 1/2$

$\mathbb{P}(C_2|O) = 1/2$

$\mathbb{P}(C_1 \cap C_2|O) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$

$\mathbb{P}(C_1|O)\mathbb{P}(C_2|O) = \mathbb{P}(C_1 \cap C_2|O)$

Yes! C_1 and C_2 are conditionally independent, conditioned on O.

Takeaway

Read a problem carefully – when we say “these steps are independent of each other” about some part of a sequential process, it’s usually “conditioned on all prior steps, these steps are conditionally independent of each other.”

Those conditional steps are usually dependent (without conditioning) because they might give you information about which branch you took.

Setting the stage: Random Variables

Implicitly defining Ω

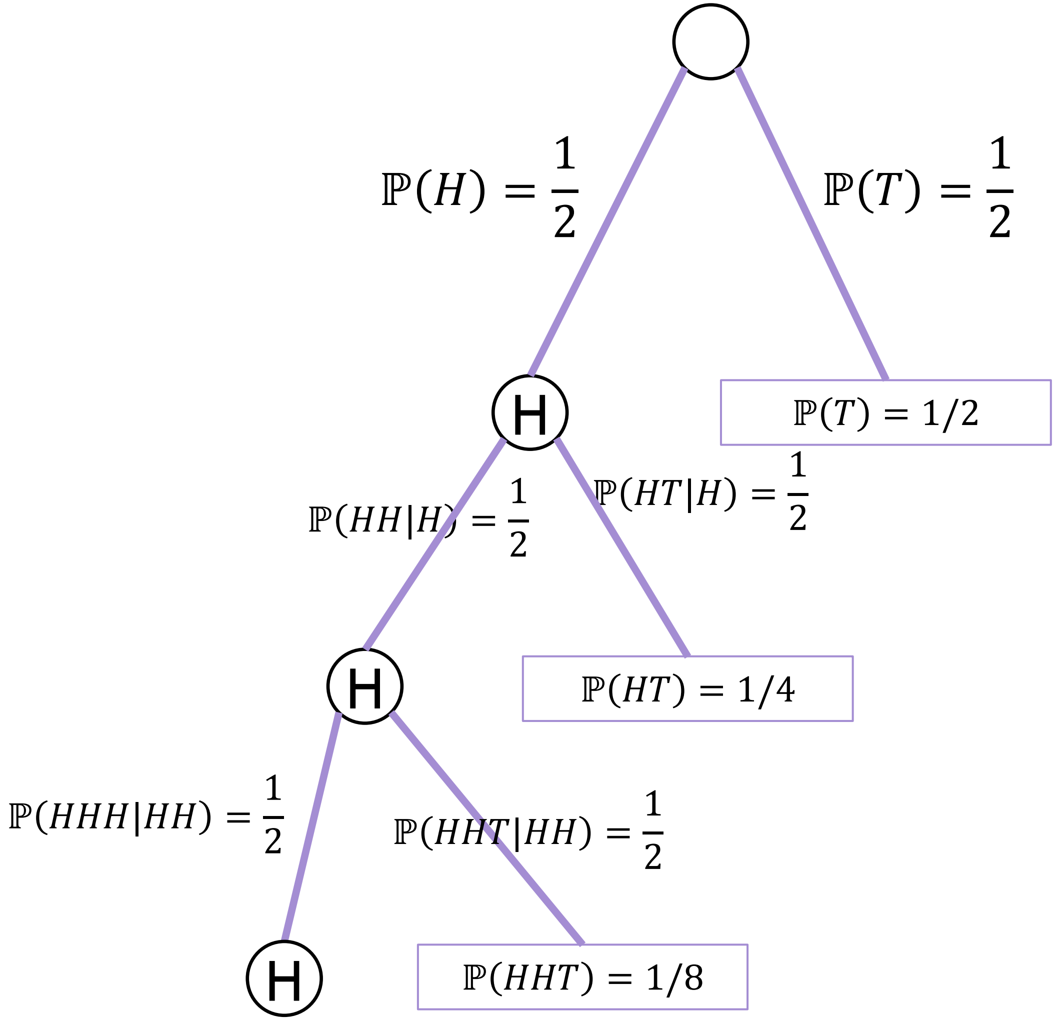
We've often skipped an explicit definition of Ω.

Often $|\Omega|$ is infinite, so we really couldn't write it out (even in principle).

How would that happen?

Flip a fair coin (independently each time) until you see your first tails.
What is the probability that you see at least 3 heads?

An infinite process.



⋮

Ω is infinite.

A sequential process is also going to be infinite…

But the tree is "self-similar"

From every node, the children look identical (H with probability ½, continue pattern; T to a leaf with probability ½)

Finding $\mathbb{P}$(at least 3 heads)

Method 1: infinite sum.

Ω includes H^iT for every i. Every such outcome has probability $\frac{1}{2^{i+1}}$

What outcomes are in our event?

$\sum_{i=3}^{\infty} \frac{1}{2^{i+1}} = \frac{\frac{1}{2^4}}{1-\frac{1}{2}} = \frac{1}{8}$

Infinite geometric series, where common ratio is between -1 and 1 has closed form $\frac{\text{first term}}{1-\text{ratio}}$

Finding $\mathbb{P}$(at least 3 heads)

Method 2:

Calculate the complement

$\mathbb{P}$(at most 2 heads) = $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$\mathbb{P}$(at least 3 heads)= $1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{1}{8}$

Random Variables

Random Variable

Often, we want to capture quantitative properties of the outcome of a random experiment.

Examples:

- What is the sum of two dice rolls?
- What is the number of coin tosses needed to see the first heads?
- What is the number of heads among 2 coin tosses?

Random Variable

Formally:

Random Variable

$X: \Omega \to \mathbb{R}$ is a random variable

$X(\omega)$ is the summary of the outcome ω

Informally: A random variable is a way to **summarize** the important (numerical) information from your outcome.

The sum of two dice

EVENTS

We could define

$E_2 =$“sum is 2”

$E_3 =$“sum is 3”

…

$E_{12} =$“sum is 12”

And ask “which event occurs”?

RANDOM VARIABLE

$X: \Omega \to \mathbb{R}$

X is the sum of the two dice.

More random variables

From one sample space, you can define many random variables.

Roll a fair red die and a fair blue die

Let D be the value of the red die minus the blue die $D(4,2) = 2$

Let S be the sum of the values of the dice $S(4,2) = 6$

Let M be the maximum of the values $M(4,2) = 4$

…

Support

The "support" (aka "the range") is the set of values X can actually take.

D (difference of red and blue) has support $\{-5, -4, -3, \dots, 4, 5\}$

S (sum) has support $\{2, 3, \dots, 12\}$

What is the support of M (max of the two dice)?

Probability Mass Function

Often, we're interested in the event $\{\omega \in \Omega : X(\omega) = x\}$

Which is the event…that $X = x$.

We'll write $\mathbb{P}(X = x)$ to describe the probability of that event

So $\mathbb{P}(S = 2) = \frac{1}{36}$, $\mathbb{P}(S = 7) = \frac{1}{6}$

The function that tells you $\mathbb{P}(X = x)$ is the "probability mass function"

We'll often write $p_X(x)$ for the pmf.

Partition

A random variable partitions Ω.

Let T be a random variable representing the number of twos in rolling a (fair) red and blue die.

$p_T(0) = 25/36$

$p_T(1) = 10/36$

$p_T(2) = 1/36$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Try It Yourself

There are 20 balls, numbered 1,2,…,20 in an urn.

You'll draw out a size-three subset. (i.e. without replacement)

$\Omega = \{$size three subsets of $\{1, \dots, 20\}$ }, $\mathbb{P}()$ is uniform measure.

Let X be the largest value among the three balls.

If outcome is $\{4,2,10\}$ then $X = 10$.

Write down the pmf of X.

Try It Yourself

There are 20 balls, numbered 1,2,…,20 in an urn.

You'll draw out a size-three subset. (i.e. without replacement)

Let X be the largest value among the three balls.

$$p_X(x) = \begin{cases} \binom{x-1}{2} / \binom{20}{3} & \text{if } x \in \mathbb{N},\ 3 \leq x \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

Good check: if you sum up $p_X(x)$ do you get 1?

Good check: is $p_X(x) \geq 0$ for all x? Is it defined for all x?

Describing a Random Variable

The most common way to describe a random variable is the PMF.

But there's a second representation:

The cumulative distribution function (CDF) gives the probability $X \leq x$

More formally, $\mathbb{P}(\{\omega: X(\omega) \leq x\})$

Often written $F_X(x) = \mathbb{P}(X \leq x)$

$F_X(x) = \sum_{i: i \leq x} p_X(i)$

Try It Yourself

What is the CDF of X where X be the largest value among the three balls? (Drawing 3 of the 20 without replacement)

Fill out the poll everywhere so Kushal knows how long to explain
Go to pollev.com/cse312su21

Try It Yourself

What is the CDF of X where X be the largest value among the three balls? (Drawing 3 of the 20 without replacement)

$$F_X(x) = \begin{cases} 0 & \text{if } x < 3 \\ \binom{\lfloor x \rfloor}{3} / \binom{20}{3} & \text{if } 3 \le x \le 20 \\ 1 & \text{otherwise} \end{cases}$$

Try It Yourself

What is the CDF of X where X be the largest value among the three balls? (Drawing 3 of the 20 without replacement)

$$F_X(x) = \begin{cases} 0 & \text{if } x < 3 \\ \binom{\lfloor x \rfloor}{3} / \binom{20}{3} & \text{if } 3 \leq x \leq 20 \\ 1 & \text{otherwise} \end{cases}$$

Good checks: Is $F_X(\infty) = 1$? If not, something is wrong.

Is $F_X(x)$ increasing? If not, something is wrong.

Is $F_X(x)$ defined for all real number inputs? If not, something is wrong.

Two descriptions

PROBABILITY MASS FUNCTION

Defined for all $\mathbb{R}$ inputs.

Usually has "0 otherwise" as an extra case.

$\sum_x p_X(x) = 1$

$0 \leq p_X(x) \leq 1$

$\sum_{z:z\leq x} p_X(z) = F_X(x)$

CUMULATIVE DISTRIBUTION FUNCTION

Defined for all $\mathbb{R}$ inputs.

Usually has "0 otherwise" and 1 otherwise" extra cases

Non-decreasing function

$0 \leq F_X(x) \leq 1$

$\lim_{x\to-\infty} F_X(x) = 0$

$\lim_{x\to\infty} F_X(x) = 1$

More Practice: Random Variables

More Random Variable Practice

Roll a fair die n times. Let X be the number of rolls that are $5s$ or $6s$.

What is the pmf?

Don't try to write the CDF…it's a mess…

Or try for a few minutes to realize it isn't nice.

More Random Variable Practice

Roll a fair die n times. Let Z be the number of rolls that are $5s$ or $6s$.

What's the probability of getting exactly z 5's/6's?

We need to know which z of the n rolls are 5's/6's. And then multiply by the probability of getting exactly that outcome

$$p_Z(z) = \begin{cases} \binom{n}{z} \cdot \left(\frac{1}{3}\right)^z \left(\frac{2}{3}\right)^{n-z} & \text{if } z \in \mathbb{Z}, 0 \leq z \leq n \\ 0 & \text{otherwise} \end{cases}$$

More Practice: Infinite sequential processes

Infinite sequential process

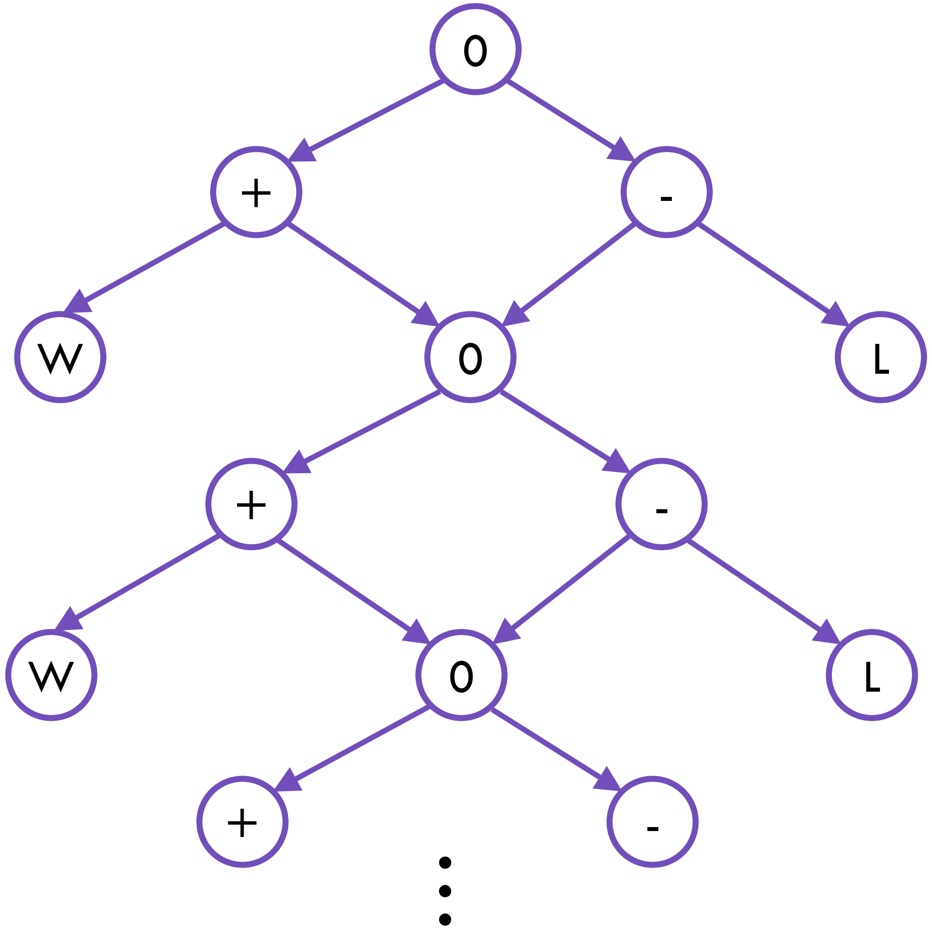
In volleyball, sets are played first team to

- Score 25 points
- Lead by at least 2

At the same time wins a set.

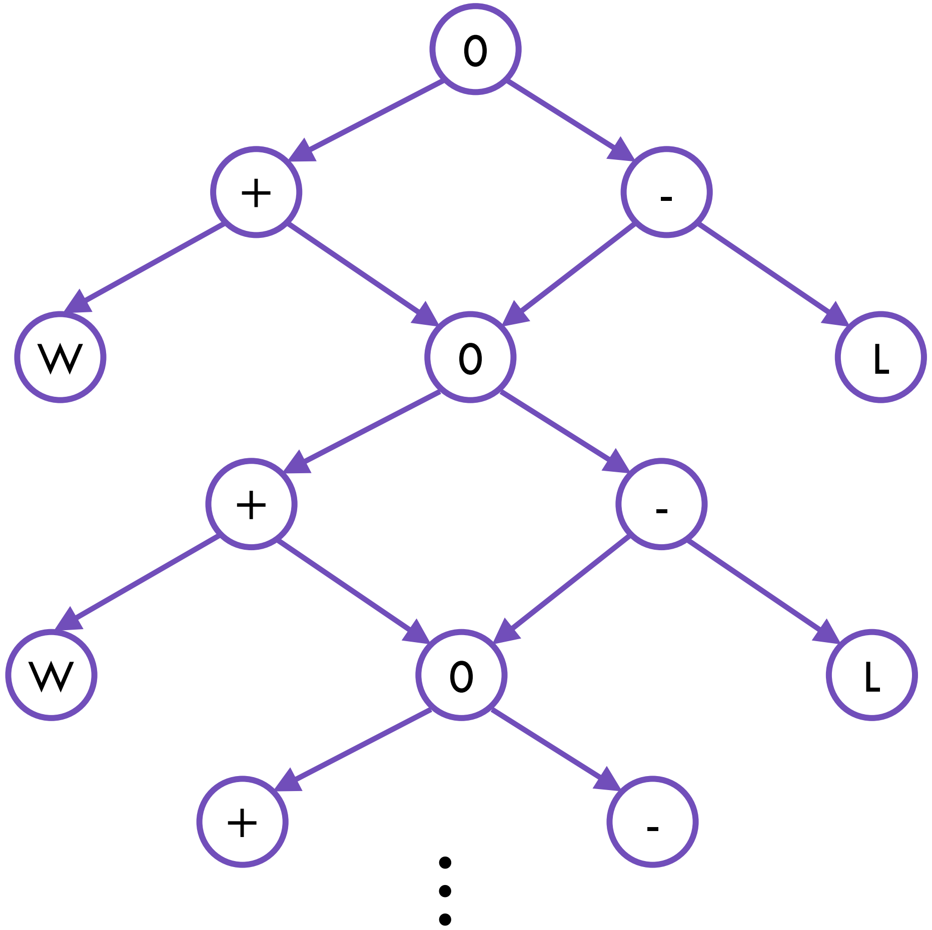
Suppose a set is 23-23. Your team wins each point independently with probability p.
What is the probability your team wins the set?

Sequential Process



$$\mathbb{P}(win\ from\ even) = p^2 + 2p(1-p)\mathbb{P}(win\ from\ even)$$

Sequential Process



$$\mathbb{P}(win\ from\ even) = p^2 + 2p(1-p)\mathbb{P}(win\ from\ even)$$

$$x - x[2p - p^2] = p^2$$
$$x[1 - 2p + p^2] = p^2$$

$$x = \frac{p^2}{p^2 - 2p + 1}$$