Please download the activity slide!

Independence

CSE 312 Summer 21
Lecture 7

Announcements

Problem Set 1 grades have been released.

Please submit a regrade request (if needed) on Gradescope within a week.

Problem Set 2 due tomorrow.

Review Summary 1 due on Friday.

Real World Mini-project out tonight.

Problem Set 3 out tomorrow.

Announcements

Problem Set 3 out tomorrow evening.

Problem Set 3 includes a programming project – using Bayes rule to do some machine learning – detecting whether emails are spam or "ham" (legitimate emails).

Longer than the programming on Problem Set1 – please get started early!

Extra resources with common difficulties on the programming project will be linked in the Problem Set 3 pdf.

Today

Chain Rule

Independence

Conditional Independence

Chain Rule

A word of caution from last lecture

$P(B \cap C) = P(B|C) \cdot P(C)$

I often see students write things like

$\mathbb{P}([A|B]|C)$

$P(\_\_ | \_\_)$

This is not a thing.

$P(A | (B \cap C)) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$

You probably want $\mathbb{P}(A|[B \cap C])$

$A|B$ isn't an event – it's describing an event **and** telling you to restrict the sample space. So, you can't ask for the probability of that conditioned on something else.

$P(A \cap B \cap C) = P(A | (B \cap C)) \cdot P(B \cap C)$

$= P(A | B \cap C) \cdot P(B|C) \cdot P(C)$

Chain Rule

We defined conditional probability as: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Which means $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$

Chain Rule

$P(A_1 | A_2 \cap A_3 \ldots A_n)$ . . . . . $P(A_{n+1} | A_n) P(A_n)$

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)$$
$$= \mathbb{P}(A_n | A_1 \cap \cdots \cap A_{n-1}) \cdot \mathbb{P}(A_{n-1} | A_1 \cap \cdots \cap A_{n-2}) \cdots \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_1)$$

$P(A_3 | A_1 \cap A_2) \longleftrightarrow P(A_1 \cap A_2)$

$P(A_1 \cap A_2 \cap A_3)$

Chain Rule Example

Shuffle a standard deck of 52 cards (so every ordering is equally likely).

Let A be the event "The top card is a K♦"

Let B be the event "the second card is a Q♥"

Let C be the event "the third card is an A♠"

What is $\mathbb{P}(A \cap B \cap C)$?

Use the chain rule!

$$\mathbb{P}(A) \cdot \mathbb{P}(B|A) \cdot \mathbb{P}(C|A \cap B)$$

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

Independence

Definition of Independence

We’ve calculated conditional probabilities.

Sometimes conditioning – getting some partial information about the outcome and restricting the sample space – doesn’t change the probability.

We already saw an example like this…

Revisiting Conditioning Practice

- Red die 6 conditioned on sum 7 1/6 $\frac{1}{6}$.
- Red die 6 conditioned on sum 9 1/4
- Sum 7 conditioned on red die 6 1/6

Red die 6 has probability 1/6 before or after conditioning on sum 7.

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Independence

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A).P(B)}{P(B)}$$

$$= P(A)$$

Independence

Two events A, B are (statistically) independent if

$$\mathbb{P}(A \cap B) = \underbrace{\mathbb{P}(A)} \cdot \underbrace{\mathbb{P}(B)}$$

You'll sometimes see this called "statistical independence" to emphasize that we're talking about probabilities (not, say, physical interactions).

If A, B both have non-zero probability then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \Leftrightarrow \mathbb{P}(A|B) = \mathbb{P}(A) \Leftrightarrow \mathbb{P}(B|A) = \mathbb{P}(B)$$

Examples

We flip a fair coin three times. Each flip is independent. (both in the statistical independence sense and in the "doesn't affect the next one" sense).

$P(E \cap F) \neq P(E) \cdot P(F)$

$0 \neq \frac{1}{8} \cdot \frac{7}{8}$

Dependent

Is $E = \{HHH\}$ independent of F ="at most two heads"?

$P(E) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ $\quad P(F) = 1 - P(E) = \frac{7}{8}$ $\quad P(E \cap F) = 0$

Are A ="the first flip is heads" and B ="the second flip is tails" independent?

$P(A) = \frac{1}{2}$ $\quad P(B) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B)$

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Examples

Is $E = \{HHH\}$ independent of $F =$"at most two heads"?

$\mathbb{P}(E \cap F) = 0$ (can't have all three heads and at most two heads).

$\mathbb{P}(E) = \frac{1}{8}$, $\mathbb{P}(F) = \frac{7}{8}$; $\mathbb{P}(E \cap F) \neq \mathbb{P}(E)\mathbb{P}(F)$.

Are $A =$"the first flip is heads" and $B =$"the second flip is tails" independent?

$\mathbb{P}(A \cap B) = \frac{2}{8}$ (uniform measure, 2 of 8 outcomes meet both A and B)

$\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}(B) = \frac{1}{2}$; $\frac{2}{8} = \frac{1}{2} \cdot \frac{1}{2}$. These are independent!

Hey Wait

I said “the flips are independent” why aren’t E, F independent?

“the flips are independent” means any event <the first flip is blah> is independent of <the second flip is blah>

But if you have an event that involves both flip one and two that might not be independent of an event involving flip one or two.

Mutual Exclusion and Independence

$A \cap B = \phi$

Two of these statements are true, one is false. Explain to each other which ones are true and find a counter-example to the false one.

1. If A, B both have nonzero probability and they are mutually exclusive, then they cannot be independent.

2. If A has zero probability, then A, B are independent (for any B).

3. If two events are independent, then at least one has nonzero probability.

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Mutual Exclusion and Independence

$A \cap B = \underbrace{\phi}$

Two of these statements are true, one is false. Explain to each other which ones are true and find a counter-example to the false one.

1. If A, B both have nonzero probability and they are mutually exclusive, then they cannot be independent.

True

$A \to$ 1 is rolled | $B \to$ 6 is rolled.

$P(A \cap B) = 0$ | $\underbrace{P(A \cap B)}_{0} = \underbrace{P(A)} \cdot \underbrace{P(B)}_{>0}$

$P(A) = P(B) = \frac{1}{6}$

2. If A has zero probability, then A, B are independent (for any B).

True

$P(A \cap B) = 0$ $\quad P(B)$

$P(A \cap B) = P(B \mid A) \cdot \boxed{P(A)} = \boxed{0}$

$P(A) = 0$

$\underbrace{P(A \cap B)}_{0} = \underbrace{P(A)}_{0} \cdot P(B)$

3. If two events are independent, then at least one has nonzero probability.

False

$P(A) = 0 \quad P(B) = 0$

$P(A \cap B) = 0$

$P(A \cap B) = P(A) \cdot P(B)$

Conditional Independence

Conditional Independence

We say A and B are conditionally independent on C if

$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$ $\qquad P(A \mid \cancel{B} \mid C)$

i.e. if you condition on C, they are independent.

Conditional Independence

Two events A, B are independent conditioned on C if $\mathbb{P}(C) \neq 0$ and
$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C) \cdot \mathbb{P}(B \mid C)$$

Conditional Independence Example

You have two coins. Coin A is fair, coin B comes up heads with probability 0.85.

You will roll a (fair) die, if the result is odd flip coin A twice (independently); if the result is even flip coin B twice (independently)

Let C_1 be the event "the first flip is heads", C_2 be the event "the second flip is heads", O be the event "the die was odd"

Are C_1 and C_2 independent? Are they independent conditioned on O?

(Unconditioned) Independence

$\frac{1}{2} \cdot \frac{1}{2} \qquad \frac{1}{2} \cdot 0.85$

$$\mathbb{P}(C_1) = \mathbb{P}(O)\mathbb{P}(C_1|O) + \mathbb{P}(\bar{O})\mathbb{P}(C_1|\bar{O})$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0.85 = 0.675$$

$\mathbb{P}(C_2) = .675$ (the same formula works)

$$\mathbb{P}(C_1)\mathbb{P}(C_2) = .675^2 = .455625$$

$$\mathbb{P}(C_1 \cap C_2) = \mathbb{P}(O)\mathbb{P}(C_1 \cap C_2|O) + \mathbb{P}(\bar{O})\mathbb{P}(C_1 \cap C_2|\bar{O})$$

$$= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot .85^2 = .48625$$

Those aren't the same! They're not independent!

$\frac{1}{2}\left(\frac{1}{2} \cdot \frac{1}{2}\right)$

Intuition: seeing a head gives you information – information that it's more likely you got the biased coin and so the next head is more likely.

Conditional Independence

$\mathbb{P}(C_1|O) = 1/2$

$\mathbb{P}(C_2|O) = 1/2$

$\mathbb{P}(C_1 \cap C_2|O) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$

$\mathbb{P}(C_1|O)\mathbb{P}(C_2|O) = \mathbb{P}(C_1 \cap C_2|O)$

Yes! C_1 and C_2 are conditionally independent, conditioned on O.

Takeaway

Read a problem carefully – when we say “these steps are independent of each other” about some part of a sequential process, it’s usually “conditioned on all prior steps, these steps are conditionally independent of each other.”

Those conditional steps are usually dependent (without conditioning) because they might give you information about which branch you took.