Please download the activity slide! :)

# Probability

CSE 312 Summer 21
Lecture 4

# Announcements

Office Hours are all up!

Please check the calendar on the course staff webpage.

Typo in 3d.

Problem Set 1 is due this Thursday.

We will also release the details of the Review Summaries assignment soon.

It will be due at the end of next week.

# Today

So far…we've done a lot of counting.

Starting today, we get to calculate probabilities!

Mostly notation and vocabulary today.

# Probability

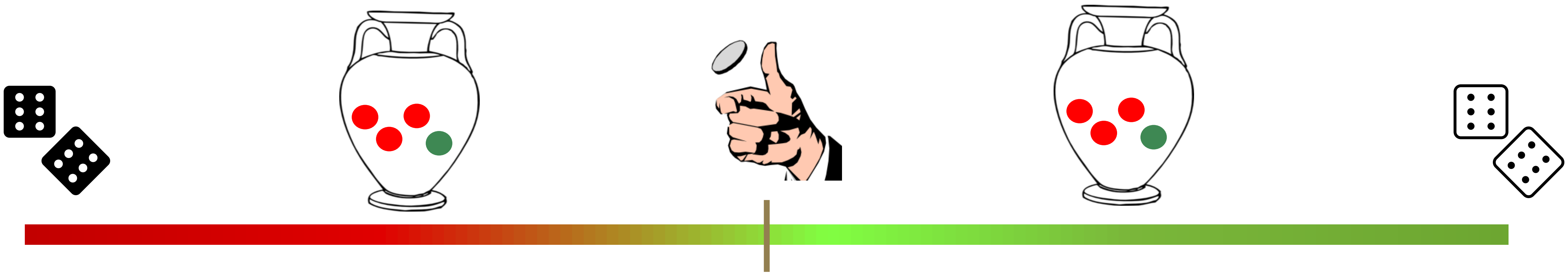
Probability is a way of **quantifying** our uncertainty.

When more than one outcome is possible,

To have “real-world” examples, we’ll need to start with some foundational processes that we’re going to assert exist

- We can flip a coin, and each face is equally likely to come up
- We can roll a die, and every number is equally likely to come up
- We can shuffle a deck of cards so that every ordering is equally likely.

# Probability Line



| | | | | |
|---|---|---|---|---|
| Rolling a seven on a fair die | Choosing a green ball | Getting Heads on a fair coin toss | Choosing a red ball | Rolling a number less than 7 on a fair die |
| Impossible event<br>0% | Unlikely event<br>> 0%; < 50% | Even Chance<br>50% | Likely event<br>> 50%; <100% | Certain event<br>100% |

# Sample Space

Omega

## Sample Space

A sample space $\Omega$ is the set of all possible outcomes of an experiment.

Examples:

For a single coin flip: $\Omega = \{H, T\}$

For a series of two coin flips: $\Omega = \{HH, HT, TH, TT\}$

For rolling a (normal) die: $\Omega = \{1,2,3,4,5,6\}$

# Events

H

## Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of $\Omega$)

Examples:

Getting a head in a single coin toss: $E = \{H\}$

Getting at least one head in two coin tosses: $E = \{HH, HT, TH\}$

Getting an even number on a die-roll: $E = \{2,4,6\}$

Getting a number less than 5 on a die-roll: $E = \{1,2,3,4\}$

# Examples

I roll a blue 4-sided die and a red 4-sided die.

The table contains the sample space.

$E_1$ The event "the sum of the dice is even" is in gold

$E_2$ The event "the blue die is 1" is in purple

| | D2=1 | D2=2 | D2=3 | D2=4 |
|---|---|---|---|---|
| D1=1 | (1,1) | (1,2) | (1,3) | (1,4) |
| D1=2 | (2,1) | (2,2) | (2,3) | (2,4) |
| D1=3 | (3,1) | (3,2) | (3,3) | (3,4) |
| D1=4 | (4,1) | (4,2) | (4,3) | (4,4) |

1 —

2 —

# Probability

# Probability

**Probability**

**A probability is a number between 0 and 1 describing how likely a particular outcome is.**

$\omega \to$ a single outcome

We will define a function $\mathbb{P}: \Omega \to [0,1]$

i.e., $\mathbb{P}$ takes an element of $\Omega$ as input and outputs the probability of the outcome which will always be between 0 and 1 (both inclusive).

We will also use $\Pr[\omega]$, $P(\omega)$ as notation.

# Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

# Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

It depends on what we want to model

If the coin is fair $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$.

But we also might have a biased coin: $\mathbb{P}(H) = .85$, $\mathbb{P}(T) = 0.15$.

# Probability Space

## Probability Space

A (discrete) probability space is a pair $(\boldsymbol{\Omega}, \mathbb{P})$ where:

$\boldsymbol{\Omega}$ is the sample space

$\mathbb{P}: \boldsymbol{\Omega} \to [\mathbf{0}, \mathbf{1}]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(\boldsymbol{x}) \geq \mathbf{0}$ for all $\boldsymbol{x} \in \Omega$
- $\sum_{\boldsymbol{x} \in \boldsymbol{\Omega}} \mathbb{P}(\boldsymbol{x}) = \mathbf{1}$

- If $\boldsymbol{E}, \boldsymbol{F} \subseteq \boldsymbol{\Omega}$ and $\boldsymbol{E} \cap \boldsymbol{F} = \emptyset$ then $\mathbb{P}(\boldsymbol{E} \cup \boldsymbol{F}) = \mathbb{P}(\boldsymbol{E}) + \mathbb{P}(\boldsymbol{F})$

# Probability Space

Flip a fair coin and roll a fair 6-sided die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Is this a valid probability space?

$\mathbb{P}$ takes in elements of $\Omega$ and outputs numbers between 0 and 1.

$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

# Measure

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

$E = \{(H,1), (H,2) \ldots (H,6)\}$

So what's the probability of seeing a heads?

Seeing heads isn't an element of the sample space!

Define $\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega)$ $= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2}$

# Probability Space

## Probability Space

A (discrete) probability space is a pair $(\Omega, \mathbb{P})$ where:

$\Omega$ is the sample space

$\mathbb{P}: \Omega \to [0,1]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(x) \geq 0$ for all $x$
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$

- If $E, F \subseteq \Omega$ and $E \cap F = \emptyset$ then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

From inclusion – exclusion $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ $\frac{0}{|\Omega|}$

# Uniform Probability Space

The most common probability measure is the uniform probability measure.

In the uniform probability measure, for every outcome $\omega$

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

die : $\frac{1}{6}$ coin : $\frac{1}{2}$

Therefore, for every event $E$

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

# Activity!

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space.

What is the probability of the event, $E$, "there are exactly 50 heads"?

A. $\frac{\binom{100}{50}}{2^{100}}$
B. $\frac{1}{101}$
C. $\frac{1}{2}$
D. $\frac{1}{2^{50}}$
E. There is not enough information in this problem.

Fill out the Poll Everywhere for Kushal to adjust his explanation
Go to pollev.com/cse312su21

$\binom{100}{1} = \frac{100!}{99!\,1!} = 100$

$\{H, T\}$

$|\Omega| = 2^{100}$

$|E| = \binom{100}{50}$

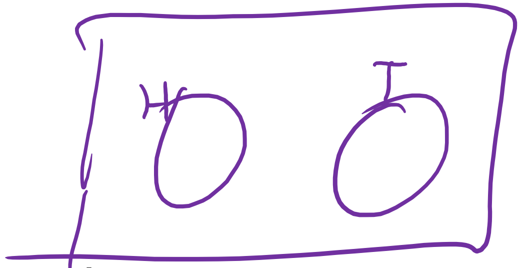
HTH

(0 Heads, 1 heads, 2 heads ... 100 heads)

$\binom{100}{1}$ ways for this to happen

$\binom{100}{50}$ ways for us to get 50 heads.

# Mutually Exclusive Events




Two events $E, F$ are mutually exclusive if they cannot happen simultaneously.

In notation, $E \cap F = \emptyset$ i.e., they are disjoint subsets of the sample space

For example, if $\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$E_1 =$ *the coin came up heads*

$E_2 =$ *the coin came up tails*

$E_3 =$ *the die showed an even number*

$E_1, E_3$ are not mutually
$E_2, E_3$ exclusive

$E_1$ and $E_2$ are mutually exclusive
$E_1$ and $E_3$ are not mutually exclusive

$E_1, E_2 \rightarrow$ Mutually Exclusive or not?

# Axioms and Consequences

We wrote down 3 requirements (axioms) on probability measures

- $\mathbb{P}(x) \geq 0$ for all $x$ (non-negativity)
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$ (normalization)
- If $E$ and $F$ are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ (countable additivity)

derived from the axioms.

These lead quickly to these three corollaries

- $\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$ (complementation)
- If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$ (monotonicity)
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$ (inclusion-exclusion)

$E \subseteq \Omega$

$\mathbb{P}(\Omega) = 1$

# More Examples!

Suppose you roll two dice. Each die is fair and they don’t affect each other. What is the probability of both dice being even?

What is your sample space?

What is your probability measure $\mathbb{P}$?

What is your event?

What is the probability?

# More Examples!

$\{(O,E), (E,E), (E,O), (O,O)\}$

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

$\{(1,1)\ (1,2)\ \ldots\ (2,1)\ \ldots\}$

What is your sample space? $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

What is your probability measure $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is your event? $\{2,4,6\} \times \{2,4,6\}$

What is the probability? $\frac{3^2}{6^2}$

$\frac{|E|}{|\Omega|} = \frac{3^2}{6^2} = \frac{1}{4}$

# More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What if we can't tell the dice apart and always put the dice in increasing order by value.

What is your sample space?

$\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6)$

$(3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)\}$

What is your probability measure $\mathbb{P}$?

$\mathbb{P}((x,y)) = 2/36$ if $x \neq y$, $\mathbb{P}(x,x) = 1/36$

What is your event? $\{(2,2), (4,4), (6,6), (2,4), (2,6), (4,6)\}$

What is the probability? $3 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} = \frac{9}{36}$ $= \frac{1}{4}$

# Takeaways

There is often more than one sample space possible! But one is probably easier than the others.

Finding a sample space that will make the uniform measure correct will probably make finding the probabilities easier to calculate.

# Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space

Probability Measure

Event

Probability

# Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: $\{(x, y): x \text{ and } y \text{ are different cards }\}$

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$

Event: all pairs with equal values

Probability: $\frac{13 \cdot P(4,2)}{52 \cdot 51}$

# Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all $52$ cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50!}{52!}$

# Takeaway

There’s often information you “don’t need” in your sample space.

It won’t give you the wrong answer.

But it sometimes makes for extra work/a harder counting problem,

Good indication: you cancelled A LOT of stuff that was common in the numerator and denominator.

# Balls and Urns

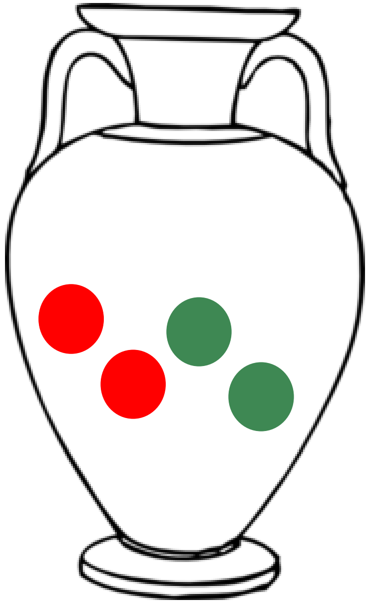
You have an urn* with two red balls and two green balls inside.

Take out two of the balls replacing the first ball after you take it out.

What's the probability of drawing out both red balls?

Sequential process: $\frac{1}{2}$ probability of the first being red

$\frac{1}{2}$ probability of the second being red.



* An urn is a vase