# Enough Counting

CSE 312 Summer 21
Lecture 3

# Announcements

Syllabus is up!

Problem Set 1 is out!

Due next Thursday 11:59 pm.

Office Hours start today!

Please start early.

# Outline

## So Far

Sum and Product Rules

Combinations and Permutations

Introduce ordering and remove it to make calculations easier

Binomial Theorem

## This Time

Principle of Inclusion-Exclusion

Pigeonhole Principle

Stars and Bars

# Principle of Inclusion-Exclusion

# Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:

- Exactly 2 'a's OR
- Exactly 1 'b' OR
- No 'x's

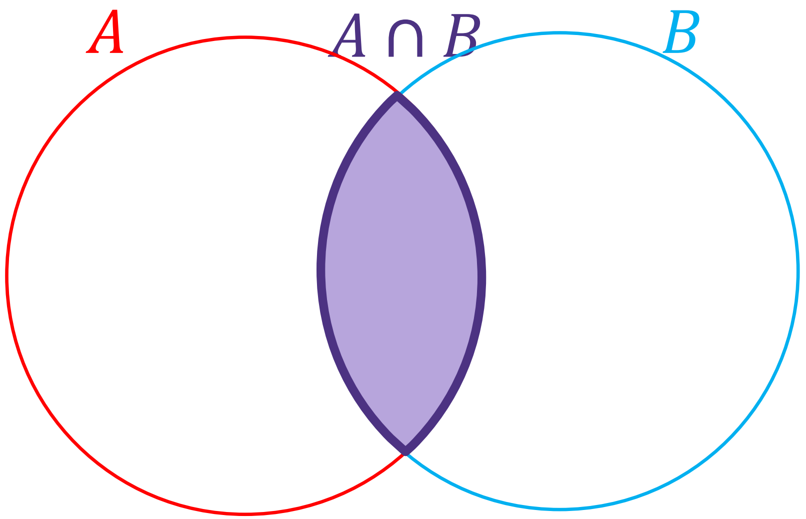
# Principle of Inclusion-Exclusion

The sum rule says when $A$ and $B$ are disjoint (no intersection), then $|A \cup B| = |A| + |B|$.

What about when $A$ and $B$ aren't disjoint?

For two sets:

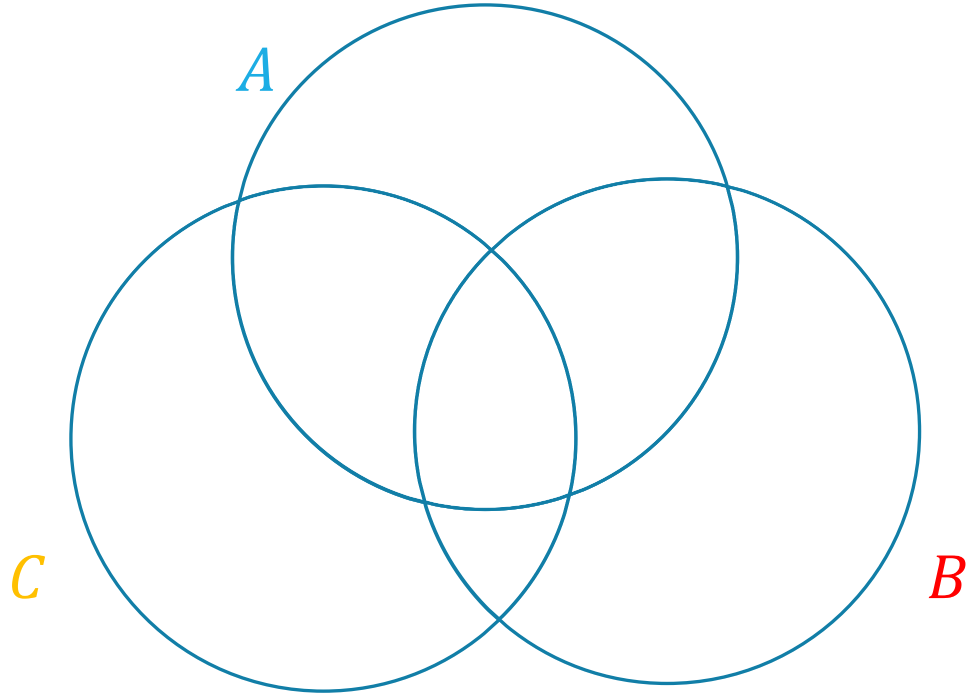
$$|A \cup B| = |A| + |B| - |A \cap B|$$



# Principle of Inclusion-Exclusion

For three sets:

$$|A \cup B \cup C| = \boxed{|A|} + \boxed{|B|} + \boxed{|C|} - \boxed{|A \cap B|} - \boxed{|B \cap C|} - \boxed{|A \cap C|} + \boxed{|A \cap B \cap C|}$$



# In general:

$$
\begin{aligned}
&|A_1 \cup A_2 \cup \cdots \cup A_n| = \\
&\quad |A_1| + |A_2| + \cdots + |A_n| \\
&\quad -(\ |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_1 \cap A_n| + |A_2 \cap A_3| + \cdots + |A_{n-1} \cap A_n|) \\
&\quad +(\ |A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|) \\
&\quad -\ \ldots \\
&+(-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|
\end{aligned}
$$

Add the individual sets, subtract all pairwise intersections, add all three-wise intersections, subtract all four-wise intersections,..., [add/subtract] the $n$-wise intersection.

# Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:

- Exactly 2 ‘a’s OR
- Exactly 1 ‘b’ OR
- No ‘x’s

For what $A, B, C$ do we want $|A \cup B \cup C|$?

# Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:
Exactly 2 ‘a’s OR
Exactly 1 ‘b’ OR
No ‘x’s

$A =$ {length 5 strings that contain exactly 2 ‘a’s}

$B =$ {length 5 strings that contain exactly 1 ‘b’s}

$C =$ {length 5 strings that contain no ‘x’s }

$|A| = \binom{5}{2} \cdot 25^3$ (need to choose which “spots” are ‘a’ and remaining string)

$|B| = \binom{5}{1} \cdot 25^4$

$|C| = 25^5$

# Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:
Exactly 2 ‘a’s OR
Exactly 1 ‘b’ OR
No ‘x’s

$A$ = {length 5 strings that contain exactly 2 ‘a’s}

$B$ = {length 5 strings that contain exactly 1 ‘b’s}

$C$ = {length 5 strings that contain no ‘x’s }

$|A| = \binom{5}{2} \cdot 25^3$
$|B| = \binom{5}{1} \cdot 25^3$
$|C| = 25^5$

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$ (choose ‘a’ spots, ‘b’ spot, remaining chars)
$|A \cap C| = \binom{5}{2} \cdot 24^3$ (choose ‘a’ spots, remaining [non-‘x’] chars)
$|B \cap C| = \binom{5}{1} \cdot 24^4$
$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$ (choose ‘a’ spots, ‘b’ spot, remaining [non-‘x’] chars)

# Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$|A| = \binom{5}{2} \cdot 25^3$

$|B| = \binom{5}{1} \cdot 25^4$

$|C| = 25^5$

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$

$|A \cap C| = \binom{5}{2} \cdot 24^3$

$|B \cap C| = \binom{5}{1} \cdot 24^4$

$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$

$$
\begin{aligned}
|A \cup B \cup C| = \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= \binom{5}{2} \cdot 25^3 + \binom{5}{1} \cdot 25^4 + 25^5 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2 - \binom{5}{2} \cdot 24^3 - \binom{5}{1} \cdot 24^4 + |A \cap B \cap C| \\
&= 11{,}875{,}000 - 1{,}814{,}400 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2 \\
&= 10{,}060{,}600 + 15{,}870 \\
&= 10{,}076{,}470
\end{aligned}
$$

# Practical tips

Give yourself clear definitions of $A, B, C$.

Make a table of all the formulas you need before you start calculating.

Calculate "size-by-size" and incorporate into the total.

Basic check: If (in an intermediate step) you ever:

1. Get a negative value
2. Get a value greater than the prior max by adding (after all the single sets)
3. Get a value less than the prior min by subtracting (after all the pairwise intersections)

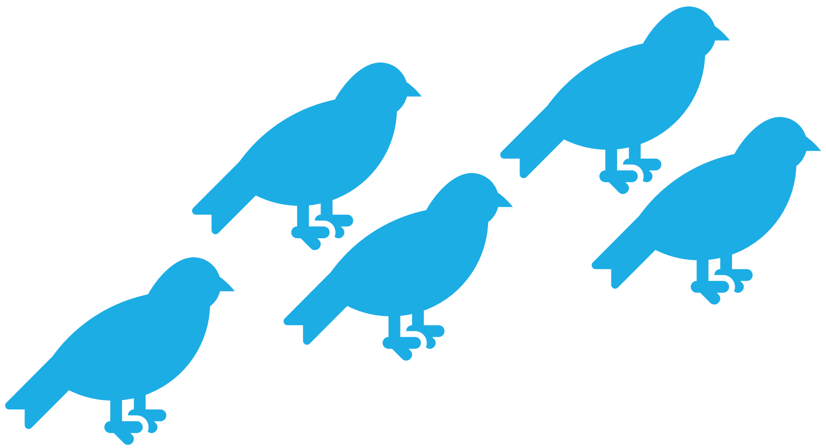
Then something has gone wrong.

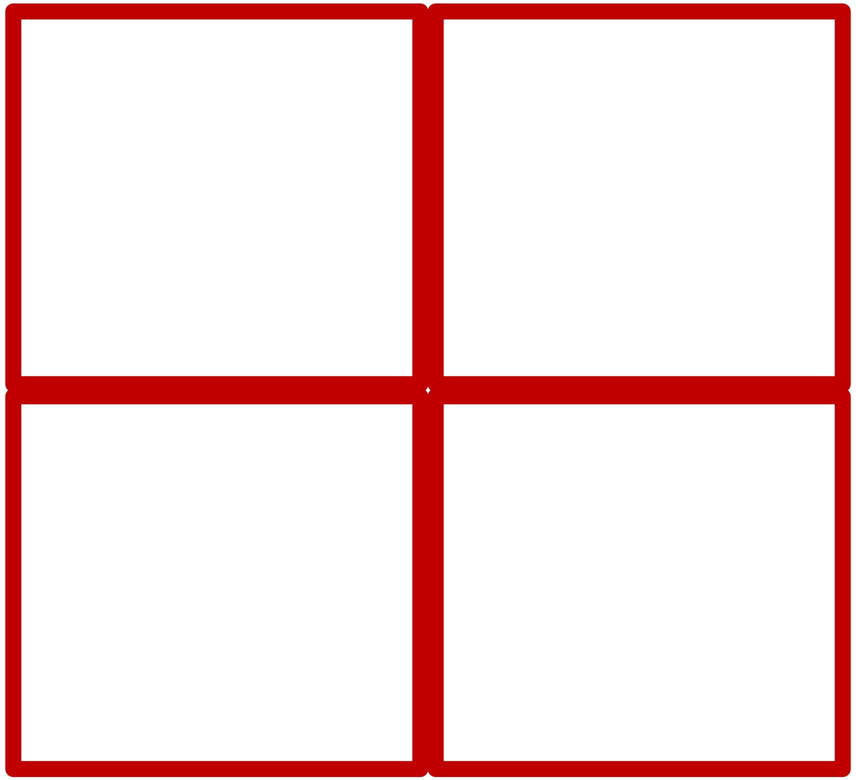
# Pigeonhole Principle

# Pigeonhole Principle

If you have $5$ pigeons, and place them into $4$ holes, then…

At least two pigeons are in the same hole.



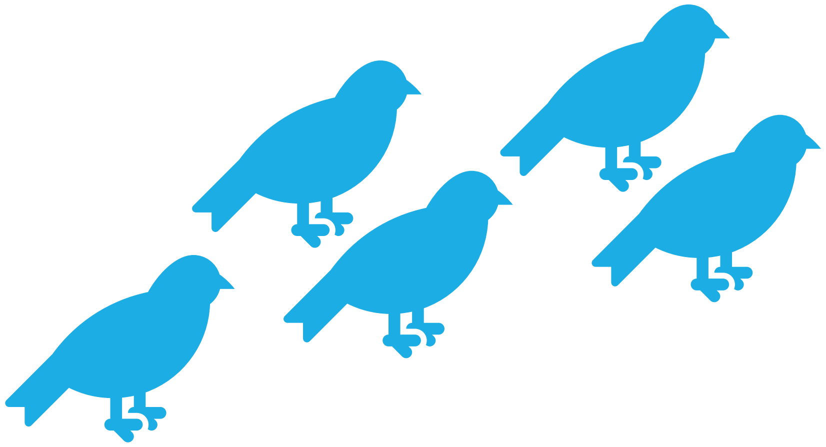


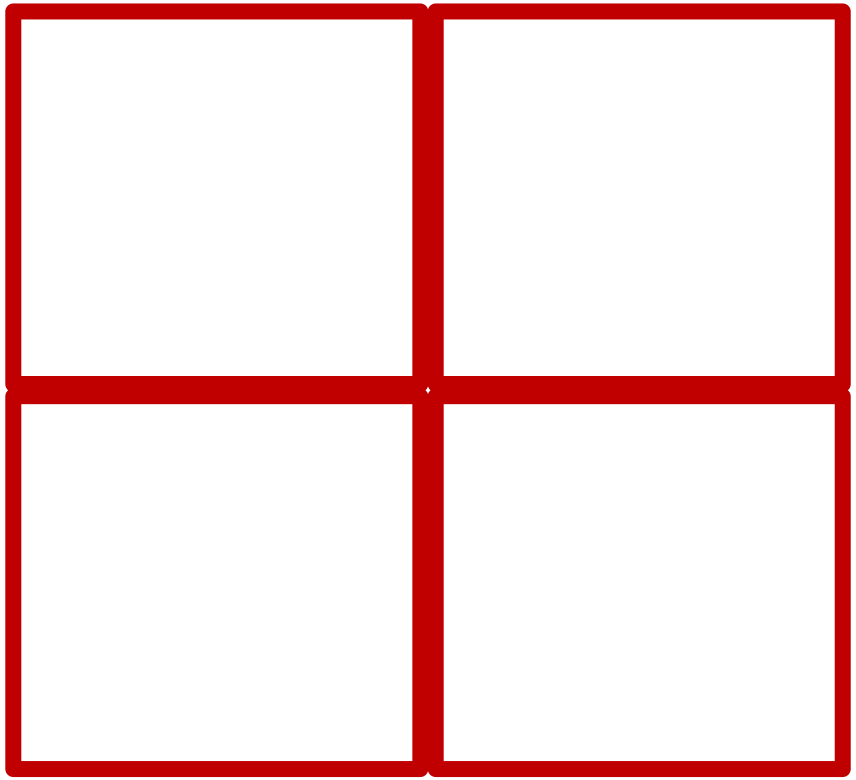
# Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

**At least** two pigeons are in the same hole.

It might be more than two.





# Strong Pigeonhole Principle

If you have $n$ pigeons and $k$ pigeonholes, then there is at least one pigeonhole that has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

$\lceil a \rceil$ is the "ceiling" of $a$ (it means always round up, $\lceil 1.1 \rceil = 2$, $\lceil 1 \rceil = 1$).

# An example

If you have to take 10 classes, and have 3 quarters to take them in, then…

Pigeons: The classes to take

Pigeonholes: The quarter

Mapping: Which class you take the quarter in.

Applying the (generalized) pigeonhole principle, there is at least one quarter where you take at least $\left\lceil \frac{10}{3} \right\rceil = 4$ courses.

# Practical Tips

When the pigeonhole principle is the right tool, it's usually the first thing you'd think of or the absolute last thing you'd think of.

When applying the principle, say:

What are the pigeons

What are the pigeonholes

How do you map from pigeons to pigeonholes

Look for – a set you're trying to divide into groups, where collisions would help you somehow.

# One Final Counting Rule

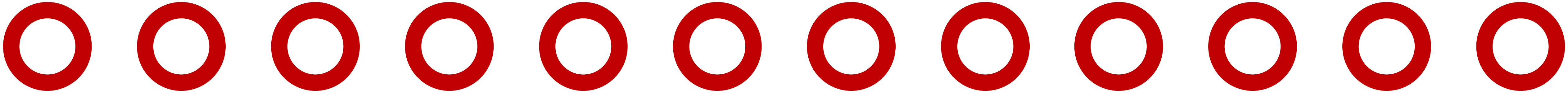
# One More Counting Rule

You’re going to buy one-dozen donuts (i.e., 12 donuts)

There are chocolate, strawberry, coconut, blueberry, and lemon (i.e. five types)

How many different donut boxes can you buy?

Consider two boxes the same if they contain the same number of every kind of donut (order doesn’t matter)



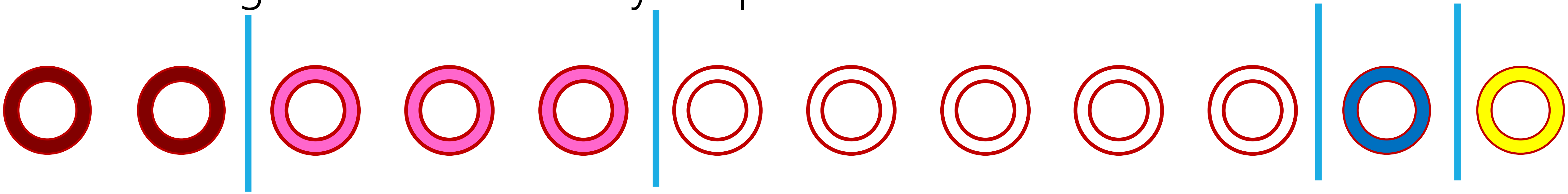
# One More Counting Rule

You're going to buy one-dozen donuts (i.e., 12 donuts)

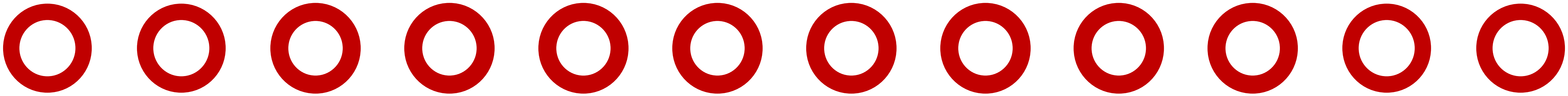
There are chocolate, strawberry, coconut, blueberry, and lemon (i.e. five types)

Put donuts in order by type, then put dividers between the types.

Counting the number of ways to place dividers instead.



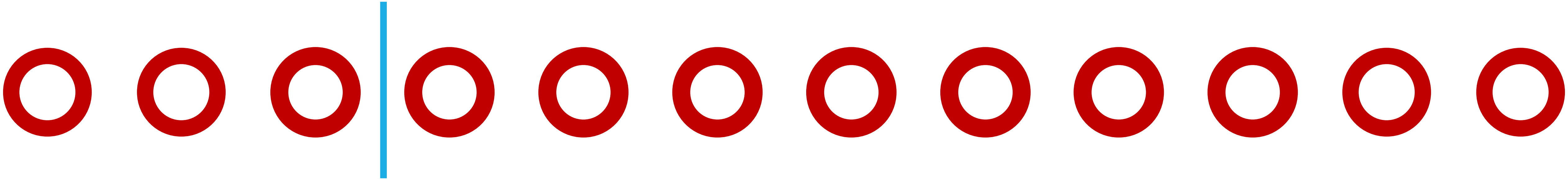
# Placing Dividers



Place a divider – how many possible locations are there?

13 – before donut 1, before 2, …, before donut 12, after donut 12.

# Placing Dividers



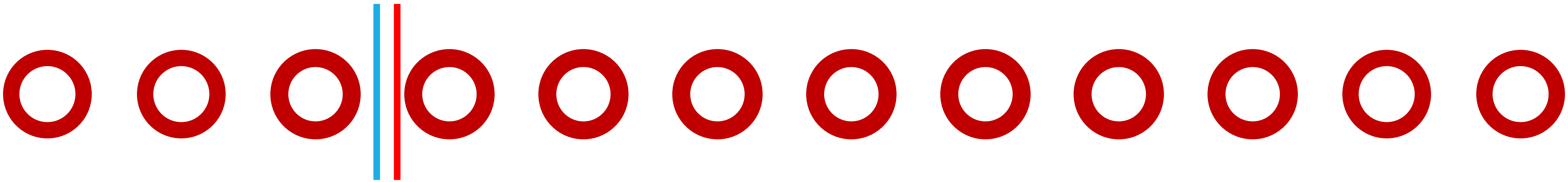
Place a divider – how many possible locations are there?

13 – before donut 1, before 2, …, before donut 12, after donut 12.

Place the second divider, how many possible locations are there?

14 – one of the previous spots was split (“before” and “after” the last divider)

# Placing Dividers



Place a divider – how many possible locations are there?

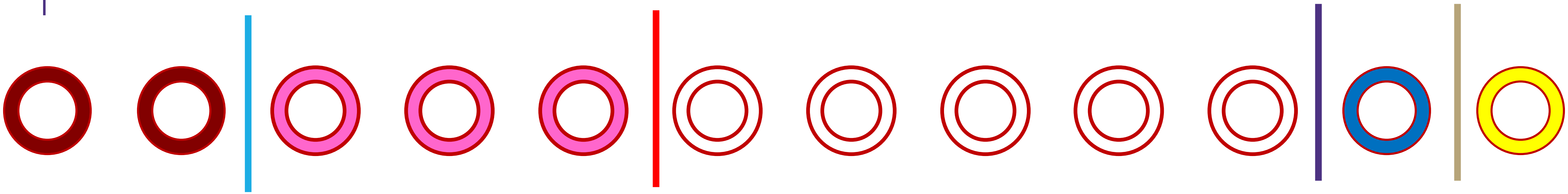
$13$ – before donut 1, before 2, …, before donut 12, after donut 12.

Place the second divider, how many possible locations are there?

$14$ – one of the previous spots was split (“before” and “after” the first divider)

In general, placing divider $i$ has $12 + i$ possible locations.

# Wrapping Up



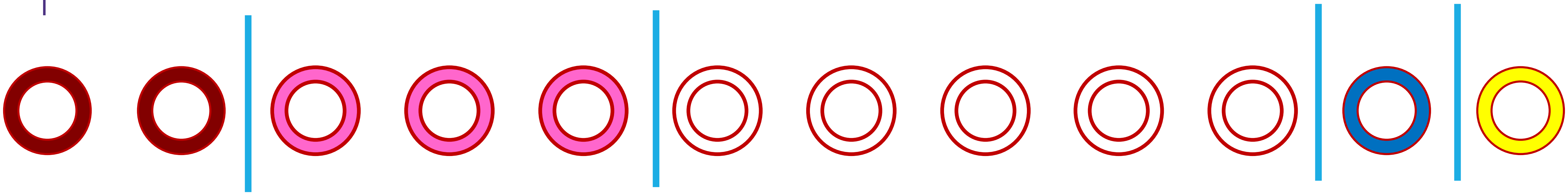
We had 12 donuts, how many dividers do we need?

4 (to divide into 5 groups)

Count so far: $13 \cdot 14 \cdot 15 \cdot 16$

Are we done?

# Wrapping Up



Count so far: $13 \cdot 14 \cdot 15 \cdot 16$

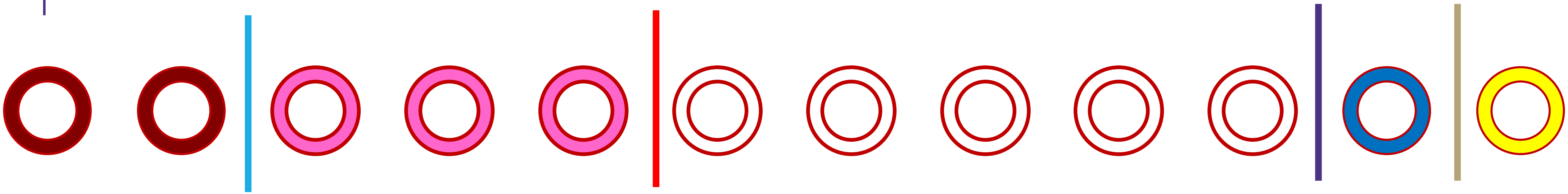
This count treats all dividers as different – they're not! Divide by $4!$.

For $n$ donuts of $k$ types

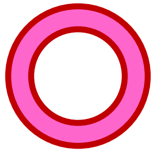
$$\frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}$$

That's a combination! $\binom{n+k-1}{k-1}$

# Wrapping Up



$\binom{n+k-1}{k-1}$

We wrote down a "string" consisting of $n$  and $k-1$ |

$n+k-1$ characters, $n$ "donuts" are identical, $k-1$ "dividers" are identical, so divide by the rearrangements (like we did for SEATTLE).

# In General

To pick $n$ objects from $k$ groups (where order doesn't matter and every element of each group is indistinguishable), use the formula:

$$\binom{n+k-1}{k-1}$$

The counting technique we did is often called "stars and bars"

using a "star" instead of a donut shape, and calling the dividers "bars"

# We've seen lots of ways to count

Sum rule (split into disjoint sets)

Product rule (use a sequential process)

Combinations (order doesn't matter)

Permutations (order does matter)

Principle of Inclusion-Exclusion

Complementary Counting

"Stars and Bars" $\binom{n+k-1}{k-1}$

Niche Rules (useful in very specific circumstances)

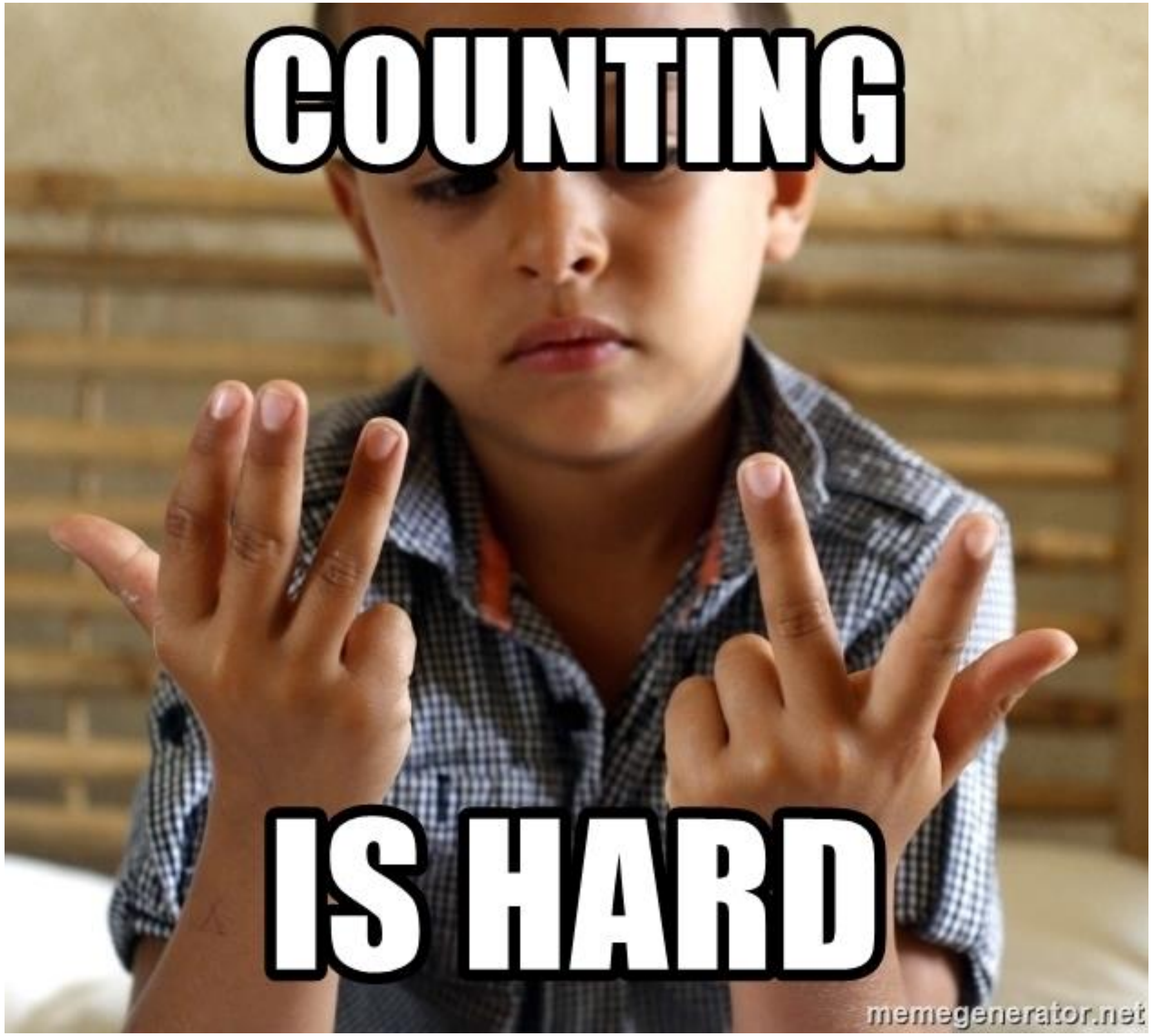
- Binomial Theorem
- Pigeonhole Principle

# Practice

How do we know which rule to apply?

PRACTICE! PRACTICE! PRACTICE!

But if as you are working you realize that things are getting out of control, put it aside and try something different.


COUNTING
IS HARD
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# Practice

# Cards

A “standard” deck of cards has 52 cards. Each card has a suit

diamonds ♦,

hearts ♥,

clubs ♧,

spades ♠

and a value (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen, King).

A “5-card-hand” is a set of 5 cards

How many five-card “flushes” are there? – a flush is a hand of cards all of the same suit.

# Cards

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit.

Way 1: How can I describe a flush? Which suit it is, and which values:

$$\binom{4}{1} \cdot \binom{13}{5}$$

# Cards

Way 2: Pretend order matters. The first card can be anything,

After that, you'll have 12 options (the remaining cards of the suit), then 11, …

Then divide by $5!$, since order isn't supposed to matter.

$$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$$

This is the same number as what we got on the last slide!

$$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!} = 4 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!} = 4 \cdot \frac{13!}{5! \cdot 8!} = \binom{4}{1} \cdot \binom{13}{5}$$

# A Solution with a Problem

**You wish to count the number of 5-card hands with at least 3 aces.**

There are 4 Aces (and 48 non aces)

$\binom{4}{3} \cdot \binom{49}{2}$

Choose the three aces. Then of the 49 remaining cards (the last ace is allowed as well, because we're allowed to have all 4)

What's wrong with this calculation?

What's the right answer?

Fill out the Poll Everywhere for Kushal to adjust his explanation
Go to pollev.com/cse312su21

# A Solution with a Problem

For a hand, there should be exactly one set of choices in the sequential process that gets us there.

{A♧, A♠, A♦} {A♥, K ♠}

And

{A♧, A♠, A♥}, {A♦, K ♠}

Are two different choices of the process, but they lead to the same hand!

# A Solution with a Problem

We could count exactly which hands appear more than once, and how many times each appears and compensate for it.

See the extra slides at the end.

An easier solution is to try again…

The problem was trying to account for the “at least” – come up with disjoint sets and count separately.

$$\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$$

If there are exactly 3 aces, we choose which 3 of the 4, then choose which 2 cards among the 48 non-aces. If all 4 aces appear, then one of the remaining 48 cards finishes the hand. Applying the sum rule completes the calculation.

# Takeaway

It’s hard to count sets where one of the conditions is “at least X”

You usually need to break those conditions up into disjoint sets and use the sum rule.

# More Practice

# Fruit Picking

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas.

You need to pick at most 2 apples and at least 1 banana. How many sets of fruit can you choose?

# Fruit Picking

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas.

You need to pick at most 2 apples and at least 1 banana. How many sets of fruit can you choose?

Divide into cases based on number of apples:

$0$ apples: $1$ to $8$ bananas possible ($8$ options)

$1$ apple: $1$ to $7$ bananas possible ($7$ options)

$2$ apples: $1$ to $6$ bananas possible ($6$ options)

$21$ total (by sum rule)

# Fruit Picking

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas.  You need to pick at most 2 apples and at least 1 banana. How many sets of fruit can you choose?

Pick out your first banana. Problem is now to pick 7 fruits (at most 2 apples, allowed to take apples oranges and bananas)

Ignore apple restriction, and subtract off when too many apples:

Ignore restriction: $\binom{7+3-1}{3-1}$

$\geq$ 3 apples, $\binom{4+3-1}{3-1}$ (choose 3 apples first, pick 4 remaining)

Total: $\binom{9}{2} - \binom{6}{2} = 36 - 15 = 21$

# Takeaways

For donut-counting style problems with “twists”, it sometimes helps to “just throw the first few in the box” to get a problem that is exactly in the donut-counting framework.

When you can do a problem two **very** different ways and get the same answer, you get much more confident in the answer.

# Fixing The Overcounting

# A Solution with a Problem

You wish to count the number of 5-card hands with at least 3 aces.

There are 4 Aces (and 48 non aces)

$\binom{4}{3} \cdot \binom{49}{2}$

Choose the three aces. Then of the 49 remaining cards (the last ace is allowed as well, because we're allowed to have all 4)

What's wrong with this calculation?

# The Problem

When do we overcount?

If there are exactly 4 Aces in the hand, then we count the hand 4 different times (once for each ace as an "extra" one):

{A♧, A♠, A♦}, {A♥, ?}

{A♧, A♠, A♥}, {A♦, ?}

{A♧, A♥, A♦}, {A♠, ?}

{A♥, A♠, A♦}, {A♧, ?}

# How much do we overcount?

There are 48 such hands (one for every card that could be "?" on the last slide)

So we've counted $3 \cdot 48$ processes that shouldn't count.

That would give a corrected total of $\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 48$

This is the same number as we got during lecture with our other counting.