CSE 312: Foundations of Computing II
Quiz Section #5: Review (solutions)

1. Let A and B be events in the same sample space that each have nonzero probability. For each of the following statements, state whether it is always true, always false, or it depends on information not given.

   (a) If A and B are mutually exclusive, then they are independent.

   False

   (b) If A and B are independent, then they are mutually exclusive.

   False

   (c) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are mutually exclusive.

   False

   (d) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are independent.

   Depends whether $\mathbb{P}(A \cap B) = 9/16$

2. Given any set of 18 integers, show that one may always choose two of them so that their difference is divisible by 17.

   By the pigeonhole principle, two of them, say x and y, must have the same remainder when divided by 17. That means $x \equiv y \pmod{17}$, which in turn means 17 divides $x - y$.

3. Consider the following inequality: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \leq 70$. A solution to this inequality over the nonnegative integers is a choice of a nonnegative integer for each of the 6 variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the inequality. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the inequality?

   This is equivalent to asking how many different solutions are there to the equation $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 70$. The answer is $\binom{76}{6}$.

4. You roll three fair dice, each with a different numbers of faces: die 1 has six faces (numbered 1 … 6), die 2 has eight faces (numbered 1 … 8), and die 3 has twelve faces (numbered 1 … 12). Let the random variable X be the sum of the three values rolled. What is $\mathbb{E}[X]$?

   Let D_1, D_2, D_3 be the values of die 1, die 2, and die 3, respectively. $\mathbb{E}[D_1] = 3.5$, $\mathbb{E}[D_2] = 4.5$, and $\mathbb{E}[D_3] = 6.5$. Therefore, $\mathbb{E}[X] = \mathbb{E}[D_1 + D_2 + D_3] = \mathbb{E}[D_1] + \mathbb{E}[D_2] + \mathbb{E}[D_3] = 3.5 + 4.5 + 6.5 = 14.5$.

5. How many integers in $\{1, 2, \ldots, 360\}$ are divisible by one or more of the numbers 2, 3, and 5?

Inclusion-exclusion:

$$\frac{360}{2}+\frac{360}{3}+\frac{360}{5}-\frac{360}{2\times 3}-\frac{360}{2\times 5}-\frac{360}{3\times 5}+\frac{360}{2\times 3\times 5}=180+120+72-60-36-24+12=264$$

6. Suppose a special deck has 4 suits with 5 cards in each suit. It is shuffled well and then dealt into 5 piles of 4 cards each. Let E_i refer to the event that pile i has exactly one spade. Compute the probability $\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5)$.

$$\frac{20\cdot 16\cdot 12\cdot 8\cdot 4}{20\cdot 19\cdot 18\cdot 17\cdot 16}\approx 0.066$$

7. You are trying to diagnose the probability that a patient with a positive blood sugar test result has diabetes, even though she is in a low risk group. The probability of a woman in this group having diabetes is 0.8%. 90% of women with diabetes will test positive in the blood sugar test. 7% of women without diabetes will test positive in the blood sugar test. Your patient tests positive in the blood sugar test. What is the probability that she has diabetes?

   Let D be the event that she has diabetes and $+$ be the event of a positive test.

$$\mathbb{P}(D \mid +)=\frac{\mathbb{P}(+ \mid D)\mathbb{P}(D)}{\mathbb{P}(+ \mid D)\mathbb{P}(D)+\mathbb{P}(+ \mid \overline{D})\mathbb{P}(\overline{D})}=\frac{0.9\times 0.008}{0.9\times 0.008+0.07\times 0.992}\approx 0.09$$

   Notice that the posterior probability 0.09 of diabetes is approximately 10 times as great as the prior probability 0.008 of diabetes, but still small.

8. A very long multiple choice exam has 4 choices for each question. Charlie has studied enough so that he knows the correct answer for 1/2 of the questions; for an additional 1/4 of the questions he can eliminate one choice and chooses randomly and uniformly among the other three, and for the remaining 1/4 of the questions he chooses randomly and uniformly among all four answers.

   As the teacher, you want to determine how many answers the student actually knows. For a randomly chosen question, if Charlie answers it correctly, what is the probability he knew the answer?

   Let C be the event that Charlie has the correct answer and K be the event that Charlie knew the answer. Then

$$\begin{aligned}\mathbb{P}(C) &= \frac{1}{2}\cdot 1+\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{4}=\frac{31}{48}\\ \mathbb{P}(K \mid C) &= \frac{\mathbb{P}(C \mid K)\mathbb{P}(K)}{\mathbb{P}(C)}=\frac{1\cdot\frac{1}{2}}{31/48}=\frac{24}{31}\end{aligned}$$

9. The space shuttle has 6 O-rings: these were involved in the Challenger disaster. When the space shuttle is launched, each O-ring has a probability of failure of 0.0137, independent of whether other O-rings fail.

   (a) What is the probability that, during 23 launches, no O-ring will fail, but that at least one O-ring will fail during the 24th launch?

   The probability that no O-ring fails on a single launch is $(1 - 0.0137)^6 \approx 0.921$. The probability that this happen for 23 launches and doesn't happen on the 24th launch is $0.921^{23}(1 - 0.921) \approx 0.0118$.

   (b) What is the probability that no O-ring fails during 24 launches?

   $0.921^{24} \approx 0.137$

10. Suppose you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur? Assume that there are 365 possible birthdays and each one is equally probable for a randomly chosen person.

    This is a geometric distribution:

$$\left(\frac{364}{365}\right)^{19} \frac{1}{365} \approx 0.0026$$

11. Two fair 6-sided dice are thrown n times in succession.

    (a) Compute the probability that double 6 (i.e., 6 on each die) appears at least once in the n throws.

    The probability that double 6 does not occur is $\left(\frac{35}{36}\right)^n$, so the probability that it occurs at least once is $1 - \left(\frac{35}{36}\right)^n$.

    (b) How large need n be to make this probability at least 1/2?

    Solving $1 - \left(\frac{35}{36}\right)^n \geq \frac{1}{2}$ gives the solution $n \geq 24.6$. Since n must be an integer, $n \geq 25$.

12. You are working on a difficult passage from a new piece you are learning on the piano. You wish to play it correctly 4 times before stopping for the day. If your probability of playing it correctly on each attempt is 2/3, and the attempts are independent (unfortunately!), what is the probability that you have to play it at least 8 times?

    This is equivalent to asking the probability that, in the first 7 attempts, you play it correctly 3 or fewer times. Let X be the number of times you play it correctly in the first 7 attempts.

Then $X \sim \text{Bin}(7, 2/3)$.

$$\mathbb{P}(X \le 3) = \binom{7}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^7 + \binom{7}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^6 + \binom{7}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^5 + \binom{7}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^4 = \frac{379}{3^7} \approx 0.173$$

13. The probability that a customer pays with cash is 40%, independent of other customers. Find the probability that the 12th customer to arrive at the cashier is the 8th one that pays with cash.

$$\binom{11}{7}(0.4)^7(0.6)^4(0.4) \approx 0.028$$

14. Let X be the outcome of rolling a fair 6-sided die once. Let Y be the sum of the outcomes of rolling the same die n times independently.

    (a) Compute $\mathbb{E}[X]$.

    $$\mathbb{E}[X] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2}$$

    (b) Compute $\text{Var}(X)$ and the standard deviation σ of X.

    $$\begin{aligned} \mathbb{E}[X^2] &= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6} \\ \text{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} \\ \sigma &= \sqrt{\frac{35}{12}} \approx 1.7 \end{aligned}$$

    (c) Compute $\mathbb{E}[Y]$.

    For $1 \le i \le n$, let X_i be the outcome of the i-th die roll.

    $$\mathbb{E}[Y] = \text{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{7}{2} = \frac{7}{2}n$$

    (d) Compute $\text{Var}(Y)$.

    Because $X_1, X_2, \ldots, X_n$ are independent,

    $$\text{Var}(Y) = \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) = \sum_{i=1}^{n} \frac{35}{12} = \frac{35}{12}n$$

15. For $n > 1$, let $a_1, ..., a_n \in [0, 1]$. Show that there exist numbers $x_1, ..., x_n \in \{-1, 0, 1\}$ not all zero such that $\left|\sum_{i=1}^n a_i x_i\right| \leq \dfrac{n}{2^n - 2}$.

First realize that, by the pigeonhole principle, for an interval of length ℓ, if you put k points in them, there must be two such that the distance between them is $\leq \dfrac{\ell}{k-1}$. To see this, split the interval $[0, \ell]$ into $k-1$ equal length subintervals: $\left[0, \dfrac{\ell}{k-1}\right], \left[\dfrac{\ell}{k-1}, \dfrac{2\ell}{k-1}\right], ..., \left[\dfrac{(k-2)\ell}{k-1}, 1\right]$, and by the pigeonhole principle, two of the k points must belong to the same interval and hence have distance between them $\leq \dfrac{\ell}{k-1}$. For each of the $2^n - 1$ nonempty subsets $I \subseteq [n]$, consider the sum $S_I = \sum_{i \in I} a_i$. Notice that $0 \leq S_I \leq n$ since each $0 \leq a_i \leq 1$. Thus, there exist nonempty subsets I and J of $[n]$ with $I \neq J$ such that $|S_I - S_J| \leq \dfrac{n}{2^n - 2}$, by applying the previous result with $\ell = n, k = 2^n - 1$. Note that $|S_I - S_J| = \left|\sum_{i \in I} a_i - \sum_{j \in J} a_j\right|$. For each $k = 1, ..., n$, define $x_k = 1$ if $a_k \in I \setminus J$, $x_k = 0$ if $a_k \in I \cap J$, and $x_k = -1$ if $a_k \in J \setminus I$. Then $|S_I - S_J| = \left|\sum_{i \in I} a_i - \sum_{j \in J} a_j\right| = \left|\sum_{i=1}^n a_i x_i\right| \leq \dfrac{n}{2^n - 2}$.

16. At a reception, n people give their hats to a hat-check person. When they leave, the hat-check person gives each of them a hat chosen at random from the hats that remain. What is the expected number of people who get their own hats back?

For $1 \leq i \leq n$, let

$$
\begin{aligned}
X_i &= \begin{cases} 1, & \text{if } i\text{-th person gets own hat back} \\ 0, & \text{otherwise} \end{cases} \\
\mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{E}[X_i] \\
&= \sum_{i=1}^n (1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0)) \\
&= \sum_{i=1}^n \mathbb{P}(i\text{-th person gets own hat back}) \\
&= \sum_{i=1}^n \frac{1}{n} = 1
\end{aligned}
$$