Normal Random Variables
Normal Random Variable

Let’s get some intuition for that density...

Is $\mathbb{E}[X] = \mu$?

Yes! Plug in $\mu - k$ and $\mu + k$ and you’ll get the same density for every $k$. The density is symmetric around $\mu$. The expectation must be $\mu$. 

$X$ is a normal (aka Gaussian) random variable with mean $\mu$ and variance $\sigma^2$ (written $X \sim \mathcal{N}(\mu, \sigma^2)$) if it has the density:

$$f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
Changing the variance

Green: \( \sigma^2 = 0.7 \)
Red: \( \sigma^2 = 1 \)
Blue: \( \sigma^2 = 2 \)
Changing the mean

Green: $\sigma^2 = .7, \mu = 0$
Purple $\sigma^2 = .7, \mu = -1$
Scaling Normals

When we scale a normal (multiplying by a constant or adding a constant) we get a normal random variable back!

If $X \sim \mathcal{N}(\mu, \sigma^2)$

Then for $Y = aX + b$, $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Normals are unique in that you get a NORMAL back.

If you multiply a binomial by $3/2$ you don’t get a binomial (it’s support isn’t even integers!)

Normals also have the property that if $X, Y$ are independent normals, then $X + Y$ is also a normal.
Normalize

To turn $X \sim \mathcal{N}(\mu, \sigma^2)$ into $Y \sim \mathcal{N}(0,1)$ you want to set

$Y = \frac{X - \mu}{\sigma}$

Why normalize?

The density is a mess. The CDF does not have a pretty closed form. But we’re going to need the CDF a lot, so...
Table of Standard Normal CDF

The way we’ll evaluate the CDF of a normal is to:
1. convert to a standard normal
2. Round the “z-score” to the hundredths place.
3. Look up the value in the table.

It’s 2021, we’re using a table?

The table makes sure we have consistent rounding rules (makes it easier for us to debug with you).
You can’t evaluate this by hand – the “z-score” can give you intuition right away.
Use the table!

We’ll use the notation $\Phi(z)$ to mean $F_X(z)$ where $X \sim \mathcal{N}(0,1)$.

Let $Y \sim \mathcal{N}(5,4)$ what is $P(Y > 9)$?

$P(Y > 9) = P\left(\frac{Y - 5}{2} > \frac{9 - 5}{2}\right)$ we’ve just written the inequality in a weird way.

$= P(X > \frac{9 - 5}{2})$ where $X$ is $\mathcal{N}(0,1)$.

$= 1 - P\left(X \leq \frac{9 - 5}{2}\right) = 1 - \Phi(2.00) = 1 - 0.97725 = .02275.$
More practice

Let $X \sim \mathcal{N}(3, 2)$.

What is the probability that $1 \leq X \leq 4$
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What is the probability that $1 \leq X \leq 4$

$$P(1 \leq X \leq 4)$$

$$= P\left(\frac{1-3}{\sqrt{2}} \leq \frac{X-3}{\sqrt{2}} \leq \frac{4-3}{\sqrt{2}}\right)$$

$$\approx P\left(-1.41 \leq \frac{X-3}{\sqrt{2}} \leq .71\right)$$

$$= \Phi(.71) - \Phi(-1.41)$$

$$= \Phi(.71) - (1 - \Phi(1.41)) = .76115 - (1 - .92073) = .68188.$$
In real life

What’s the probability of being at most two standard deviations from the mean?

\[ = \Phi(2) - \Phi(-2) \]
\[ = \Phi(2) - (1 - \Phi(2)) \]
\[ = .97725 - (1 - .97725) = .9545 \]

You’ll sometimes hear statisticians refer to the “68-95-99.7 rule” which is the probability of being within 1, 2, or 3 standard deviations of the mean.