Linearity of Expectation

CSE 312 Spring 21
Lecture 12

Expectation

Expectation

The "expectation" (or "expected value") of a random variable X is:

$$\mathbb{E}[X] = \sum_k k \cdot \mathbb{P}(X = k)$$

Intuition: The weighted average of values X could take on.

Weighted by the probability you actually see them.

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[X+Y]=\mathbb{E}[X]+\mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Extending this to n random variables, $X_1, X_2, \ldots, X_n$

$$\mathbb{E}[X_1+X_2+\cdots+X_n]=\mathbb{E}[X_1]+\mathbb{E}[X_2]+\cdots+\mathbb{E}[X_n]$$

This can be proven by induction. $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$

Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}] = \mathbb{E}[\boldsymbol{X}] + \mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

$$\begin{aligned}
\mathbb{E}[\overset{a}{\boldsymbol{X}} + \overset{b}{\boldsymbol{Y}}] &= \Sigma_\omega P(\omega)(X(\omega) + Y(\omega)) \\
&= \underbrace{\Sigma_\omega P(\omega) X(\omega)} + \underbrace{\Sigma_\omega P(\omega) Y(\omega)} \\
&= \mathbb{E}[\boldsymbol{X}] + \mathbb{E}[\boldsymbol{Y}] \\
&= a\,E[X] + b\,E[Y]
\end{aligned}$$

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y:

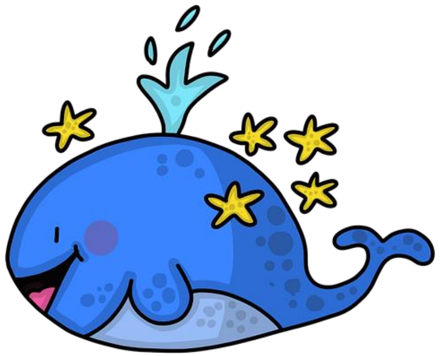
$$\mathbb{E}[\boldsymbol{X} + \boldsymbol{Y}] = \mathbb{E}[\boldsymbol{X}] + \mathbb{E}[\boldsymbol{Y}]$$

More generally, for random variables X and Y and scalars a, b and c :

$$\mathbb{E}[a\boldsymbol{X} + b\boldsymbol{Y} + c] = a\mathbb{E}[\boldsymbol{X}] + b\mathbb{E}[\boldsymbol{Y}] + c$$

$\mathbb{E}[k] = k$

Fishy Business



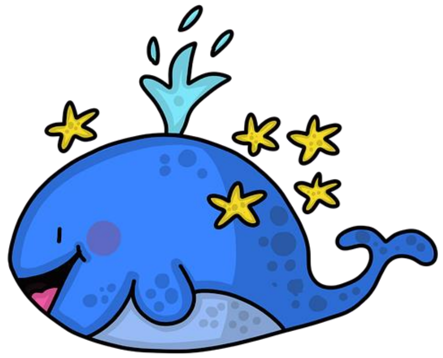
Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$ 10
- How many fish do both of you bring on an average day?

Z as the r.v. representing # of fish you both bring in

$E[Z] = E[X+Y] = E[X] + E[Y] = 3+7 = \underline{10}$

Fishy Business



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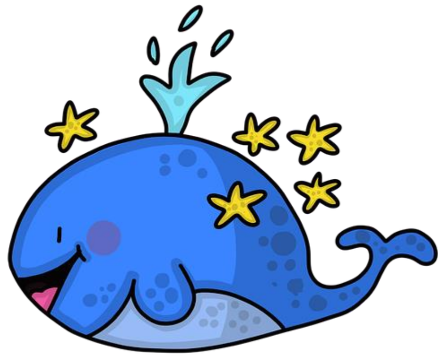
Fishy Business



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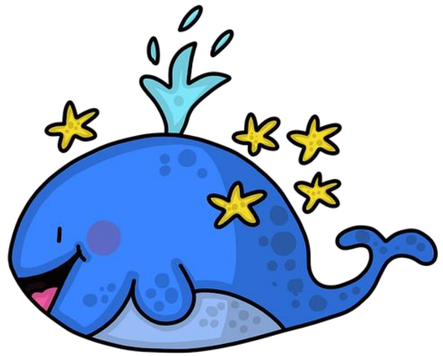
$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

- You can sell each for $10, but you need $15 for expenses. What is your average profit?

$$E[10 \cdot Z - 15] = 10\,E[Z] - 15 = 10 \cdot 10 - 15 = 100 - 15 = \underline{85}$$

Fishy Business



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- You can sell each for \$10, but you need \$15 for expenses. What is your average profit?

$$\mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15 = 100 - 15 = 85$$

Coin Tosses

If we flip a coin twice, what is the expected number of heads that come up?

$X \rightarrow$ # of heads that come up.

$$p_X(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{2} & x = 1 \\ \frac{1}{4} & x = 2 \end{cases}$$

$$E[X] = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

Coin Tosses

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$$\mathbb{E}[\boldsymbol{X}] = \Sigma_\omega P(\omega)X(\omega) = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = \mathbf{1}$$

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin **n** times?

If Y is the r.v. representing the total number of heads that come up.

$$\mathbb{E}[Y] = \sum_{k=0}^{n} k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Repeated Coin Tosses

Now what if the probability of flipping a heads was ***p*** and that we wanted to find the total number of heads flipped when we flip the coin **n** times?

$$\mathbb{E}[Y] = \sum_{k=0}^{n} k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \qquad \left[k\binom{n}{k} = n\binom{n-1}{k-1}\right]$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^{i} (1-p)^{n-1-i}$$

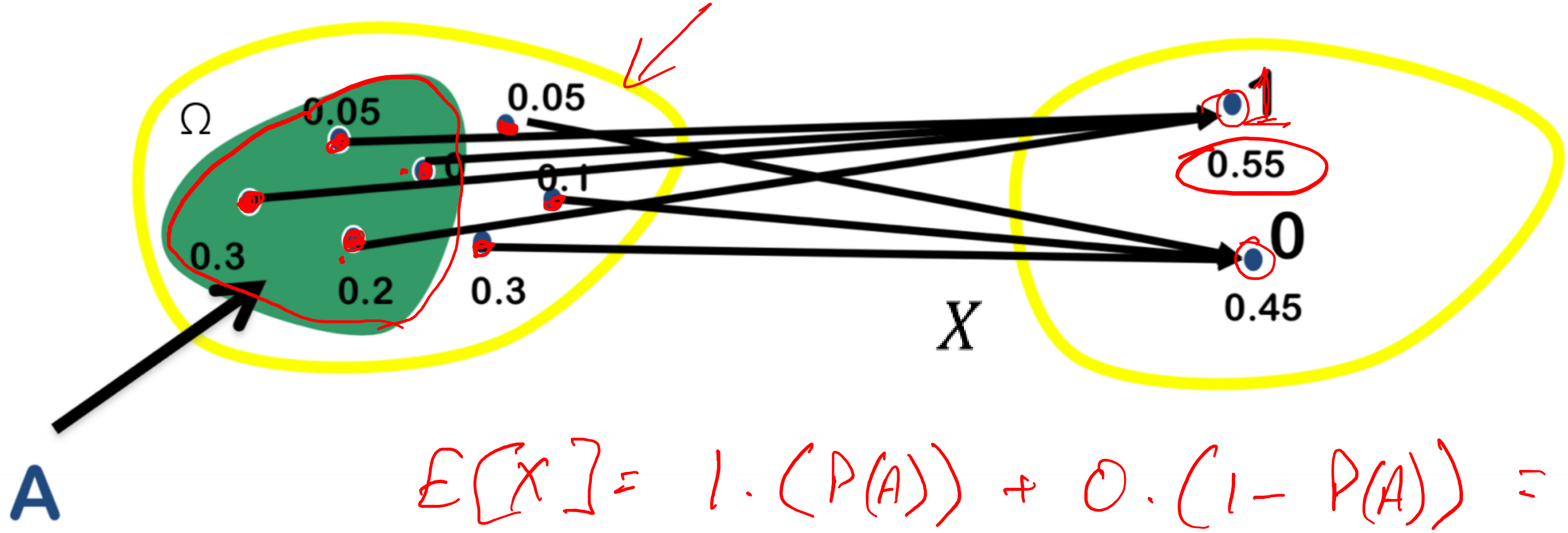
$$= np(p + (1-p))^{n-1} = \boldsymbol{np}$$

Indicator Random Variables

For any event A, we can define the indicator random variable X for A

$$X = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X = 1) = \mathbb{P}(A)$$
$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(A)$$



$E[X] = 1 \cdot (P(A)) + 0 \cdot (1 - P(A)) = P(A)$

Repeated Coin Tosses (contd)

The probability of flipping a heads is ***p*** and we wanted to find the total number of heads flipped when we flip the coin **n** times?

$X \to$ # of head flipped

$X_i \begin{cases} 1 & \text{if the } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$

$X = X_1 + X_2 + \dots + X_n$

$1 + 0 + 0 + 0 \dots + 0 = 1$

$E[X] = E[\sum_{i=1}^{n} X_i]$

$= \sum_{i=1}^{n} E[X_i]$

$\boxed{E[X_i] = p}$

$= \sum_{i=1}^{n} p$

$= np$

Repeated Coin Tosses (contd)

The probability of flipping a heads is ***p*** and we wanted to find the total number of heads flipped when we flip the coin **n** times?

Let X be the total number of heads

Let us define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if the ith coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X_i = 1) = p$$
$$\mathbb{P}(X_i = 0) = 1 - p$$

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p)$$

Repeated Coin Tosses (contd)

The probability of flipping a heads is ***p*** and we wanted to find the total number of heads flipped when we flip the coin **n** times?

Let X be the total number of heads

Let us define X_i as follows:

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$$\mathbb{P}(X_i = 1) = p$$
$$\mathbb{P}(X_i = 0) = 1 - p$$

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] = \boldsymbol{np}$$

Computing complicated expectations

We often use these three steps to solve complicated expectations

1. **Decompose**: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + X_2 + \cdots + X_n$$

2. **LOE**: Apply Linearity of Expectation

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

3. **Conquer**: Compute the expectation of each X_i , $E[X_i]$

Often X_i are indicator random variables

Pairs with the same birthday $\binom{m}{2}$

In a class of m students, on average how many pairs of people have the same birthday?

$X \rightarrow$ # of pairs of students with the same birthday

Decompose:

$$X_{ij} = \begin{cases} 1 & \text{if students } i \,\&\, j \text{ have the same birthday} \\ 0 & \text{otherwise} \end{cases}$$

$$\left[1 - \binom{m}{2}\right]$$

LOE:

$$X = \sum_{i,j=1}^{\binom{m}{2}} X_{ij} \qquad i \neq j$$

$$E[X] = E\left[\sum_{ij} X_{ij}\right] = \sum_{ij} E[X_{ij}]$$

Conquer:

$$E[X_{ij}] = P(X_{ij} = 1) = \frac{365 \quad 1}{365 \cdot 365} = \frac{1}{365}$$

$$= \binom{m}{2} \frac{1}{365}$$

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose: Let us define X as the number of pairs with the same birthday

Let us define X_{ij} as follows:

$$X_{ij} = \begin{cases} 1 & \text{if the i, j have the same birthday} \\ 0 & \text{otherwise} \end{cases} \qquad X = \Sigma_{i,j}^{\binom{m}{2}} X_{ij}$$

LOE:

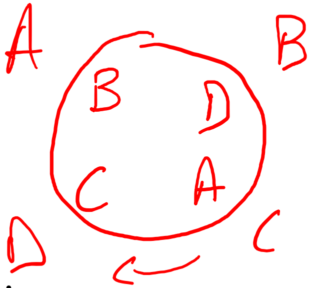
$$\mathbb{E}[X] = \Sigma_{i,j}^{\binom{m}{2}} \mathbb{E}[X_{ij}]$$

Conquer:

$$\mathbb{E}[X_{ij}] = P(X_{ij} = 1) = \frac{365}{365 \cdot 365} = \frac{1}{365}$$

$$\mathbb{E}[X] = \binom{m}{2} \cdot \mathbb{E}[X_{ij}] = \binom{m}{2} \cdot \frac{1}{365}$$

Rotating the table



n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and n-1 (equally likely)

X is the number of people that end up in front of their own name tag. **Find** $\mathbb{E}[X]$.

A	B	C	D	
C	B	D	A	1/3 – 1
A	C	B	D	1/3 – 2
D	A	C	B	1/3 – 3

Decompose:

$i = 1 \ldots n$

$$X_i = \begin{cases} 1 & \text{if the } i\text{th person end up in front of their own name tag} \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i \quad \left(1/3\right)$$

LOE:

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n-1} = \frac{n}{n-1}$$

Conquer:

$$E[X_i] = P(X_i = 1) = \frac{1}{n-1}$$

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and n-1 (equally likely)

X is the number of people that end up in front of their own name tag. **Find $\mathbb{E}[X]$.**

<u>Decompose:</u> Let us define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person i sits infront of their own name tag} \\ 0 & \text{otherwise} \end{cases} \qquad X = \Sigma_{i=1}^{n} X_i$$

<u>LOE:</u>

$$\mathbb{E}[X] = \Sigma_{i=1}^{n} \mathbb{E}[X_i]$$

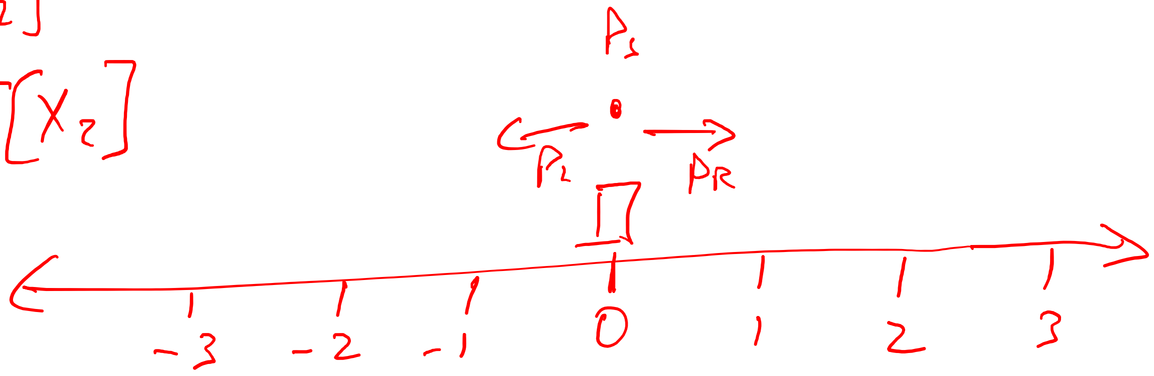
<u>Conquer:</u>

$$\mathbb{E}[X_i] = P(X_i = 1) = \frac{1}{n-1} \qquad \mathbb{E}[X] = n \cdot \mathbb{E}[X_i] = \frac{n}{n-1}$$

Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_{1} to the left with probability p_{2} and doesn't move with probability p_{3} where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let X be the location of the frog. **Find** $\mathbb{E}[X]$.

$$E[X] = E[X_1 + X_2]$$
$$= E[X_1] + E[X_2]$$
$$= 2(P_R - P_L)$$



$$X = X_1 + X_2$$

$$E[X_i] = 1 \cdot P_R + 0 \cdot P_S + (-1) P_L = (P_R - P_L)$$

$$X_i = \begin{cases} 1 & \text{if at } i \text{ frog moves right} \\ -1 & \text{" " " " " left} \\ 0 & \text{otherwise} \end{cases}$$

Frogger – Brute Force

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. **Find** $\mathbb{E}[X]$**.**

$$p_X(x) = \begin{cases} p_L^2 & x = -2 \\ 2p_L p_S & x = -1 \\ 2p_L p_R + p_S^2 & x = 0 \\ 2p_R p_S & x = 1 \\ p_R^2 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

100 / 1000

$$\mathbb{E}[\boldsymbol{X}] = \Sigma_\omega P(\omega)X(\omega) = (-2)p_L^2 + (-1)2p_L p_S + 0 \cdot \left(2p_L p_R + p_S^2\right) + (1)2p_R p_S + (2)p_R^2 = \boldsymbol{2(p_R - p_L)}$$

Frogger – LOE

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. **Find $\mathbb{E}[X]$.**

Let us define X_i as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{2} X_i\right] = \sum_{i=1}^{2} \mathbb{E}[X_i] = \mathbf{2(p_R - p_L)}$$