Section 1: Combinatorics

Review of Main Concepts (Counting)

• **Sum Rule**: If an experiment can either end up being one of \( N \) outcomes, or one of \( M \) outcomes (where there is no overlap), then the total number of possible outcomes is \( N + M \).

• **Product Rule**: Suppose events \( A_1, \ldots, A_n \) each have \( m_1, \ldots, m_n \) possible outcomes, respectively. Then there are
\[
 m_1 \cdot m_2 \cdot m_3 \cdots m_n = \prod_{i=1}^{n} m_i \text{ possible outcomes overall.}
\]

• **Number of ways to order} \( n \) distinct objects**: \( n! = n \cdot (n - 1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 \)

• **Number of ways to select from} \( n \) distinct objects**:
  
  – **Permutations** (number of ways to linearly arrange \( k \) objects out of \( n \) distinct objects, when the order of the \( k \) objects matters):

  
  \[
  P(n, k) = \frac{n!}{(n-k)!}
  \]

  – **Combinations** (number of ways to choose \( k \) objects out of \( n \) distinct objects, when the order of the \( k \) objects does not matter):

  
  \[
  \frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n, k)
  \]

1. **Seating**

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if …

(a) …all couples are to get adjacent seats?

(b) …anyone can sit anywhere, except that one couple insists on not sitting in adjacent seats?

2. **Weird Card Game**

In how many ways can a pack of fifty-two cards be dealt to thirteen players, four to each, so that every player has one card of each suit?

3. **HBCDEFGA**

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

4. **Escape the Professor**

There are 6 security professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are chosen and paired off, how many pairings are possible?
5. **Birthday Cake**

A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

6. **Full Class**

There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

7. **Paired Finals**

Suppose you are to take a CSE 312 final in pairs. There are 100 students in the class and 8 TAs, so 8 lucky students will get to pair up with a TA. Each TA must take the exam with some student, but two TAs cannot take the exam together. How many ways can they pair up?

8. **Photographs**

Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

9. **Rabbits!**

Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

10. **Extended Family Portrait**

A group of $n$ families, each with $m$ members, are to be lined up for a photograph. In how many ways can the $nm$ people be arranged if members of a family must stay together?

11. **Subsubset**

Let $[n] = \{1, 2, ..., n\}$ denote the first $n$ natural numbers. How many (ordered) pairs of subsets $(A, B)$ are there such that $A \subseteq B \subseteq [n]$?

12. **Divide Me**

How many numbers in $[360]$ are divisible by:

(a) 4, 6, and 9?

(b) 4, 6, or 9?

(c) Neither 4, 6, nor 9?