CSE 312

Foundations of Computing II

Lecture 9: Linearity of Expectation

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Last Class:

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Fn (CDF)
- Expectation

Today:

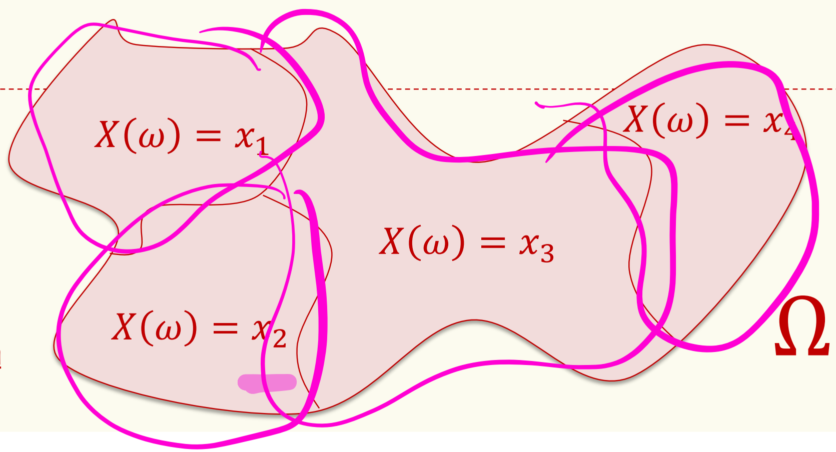
- Recap
- Linearity of Expectation
- Indicator Random Variables

Kandinsky

Reminder: Random Variables

Definition. A **random variable (RV)** defined on a probability space $(\Omega, \mathbb{P})$ is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is called its range/support Ω_X



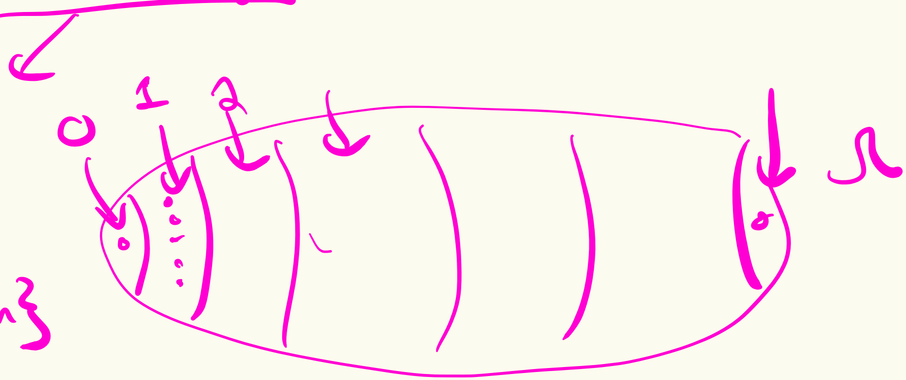
$\{X = x_i\} \stackrel{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x_i\}$

Random variables partition the sample space.

Coin flipping again

Suppose we flip a coin independently n times with probability p of coming up Heads each time. Let the r.v. Z be the number of Heads in the n coin flips.

Ω = {seqs of H/Ts of length n} $|\Omega| = 2^n$



Ω_Z : $\{0, 1, 2, \ldots, n\}$

Probability Mass Function (pmf) and Cumulative Distribution Function (CDF)

Definitions.

For a RV $X : \Omega \to \mathbb{R}$, the probability mass function (pmf) of X specifies for any real number x, the probability that $X = x$.

$$\mathrm{p}_X(x) = \Pr(X = x) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\sum_{x \in \Omega_X} \mathbb{P}(X = x) = 1$$

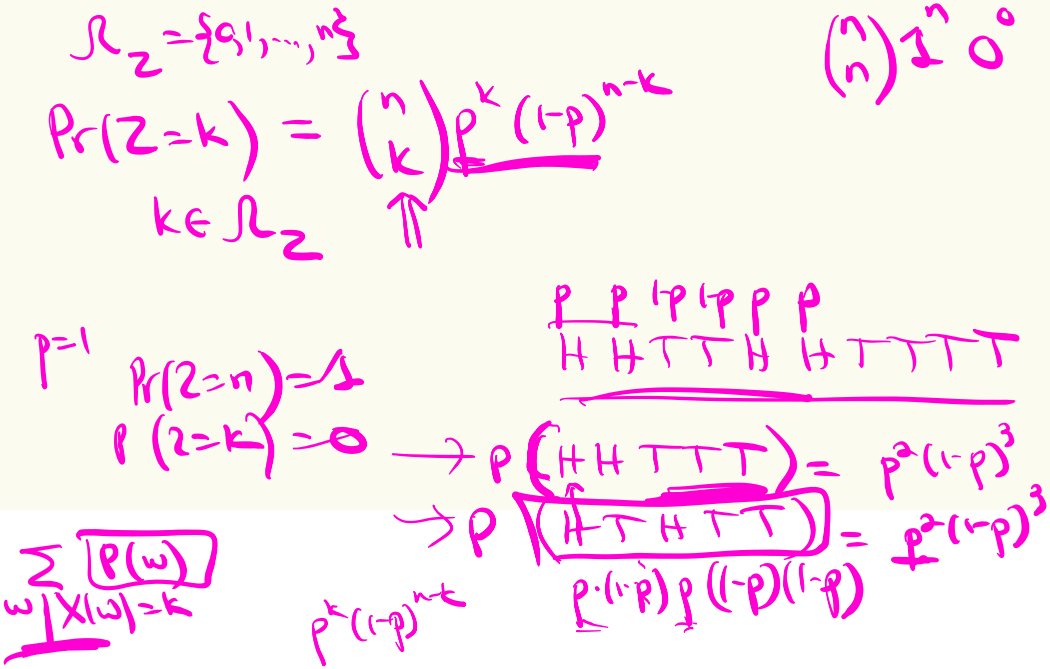
For a RV $X : \Omega \to \mathbb{R}$, the cumulative distribution function of X specifies for any real number x, the probability that $X \leq x$.

$$\mathrm{F}_X(x) = \Pr(X \leq x)$$

$$F_X(-\infty) = 0 \qquad F_X(+\infty) = 1$$

Coin flipping again

Suppose we flip a coin independently n times with probability p of coming up Heads each time. Let the r.v. Z be the number of Heads in the n coin flips. What is the p.m.f. of Z ?



Expectation of Random Variable

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the expectation or expected value of X is

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$$

or equivalently

$$E[X] = \sum_{x \in \Omega_X} x \cdot \Pr(X = x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

100 $\frac{1}{4}$

Coin flipping again

Suppose we flip a coin independently n times with probability p of coming up Heads each time. Let the r.v. Z be the number of Heads in the n coin flips. What is the $\mathbb{E}(Z)$?

$$E(Z) = \sum_{k=0}^{n} k \, P(X=k)$$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \overset{?}{=} np$$

The brute force method

we flip n coins, each one heads with probability p,

Z is the number of heads, what is $\mathbb{E}(Z)$?

$$\mathbb{E}[Z] = \sum_{k=0}^{n} k \cdot P(Z = k) \quad = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

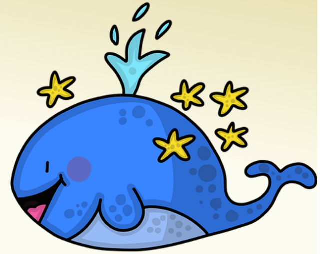
$$= \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \quad = \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^k (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np\big(p + (1-p)\big)^{n-1} = np \cdot 1 = np$$

Linearity of Expectation (Idea)



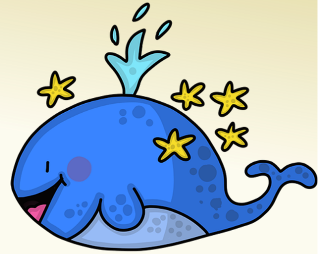
Let's say you and your friend sell fish for a living.

- Every day you catch **X** fish, with **E[X] = 3**.
- Every day your friend catches **Y** fish, with **E[Y] = 7**.

How many fish do the two of you bring in (**Z = X + Y**) on an average day?

$$E[Z] = E[X + Y] = E[X] + E[Y] = 3 + 7 = 10$$

Linearity of Expectation (Idea)



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$$E[Z] = E[X + Y] = E[X] + E[Y] = 3 + 7 = 10$$

You can sell each fish for \$5 at a store, but you need to pay \$20 in rent. How much profit do you expect to make?

$$E[5Z - 20] = 5E[Z] - 20 = 5 \times 10 - 20 = 30$$

Linearity of Expectation – Proof

defined on same prob space.

Theorem. For **any** two random variables X and Y

$$\mathbb{E}(X+Y)=\mathbb{E}(X)+\mathbb{E}(Y).$$

$Z = X + Y$

$\omega \in \Omega$

$Z(\omega) = X(\omega) + Y(\omega)$

$$\mathbb{E}(X+Y)=\sum_{\omega}P(\omega)(X(\omega)+Y(\omega))$$
$$=\sum_{\omega}P(\omega)X(\omega)+\sum_{\omega}P(\omega)Y(\omega)$$
$$=\mathbb{E}(X)+\mathbb{E}(Y)$$

Linearity of Expectation

Theorem. For **any** two random variables X and Y

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

induction

Or, more generally: For any random variables $X_1, \dots, X_n$,

$$\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n).$$

Because: $\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}((X_1 + \cdots + X_{n-1}) + X_n)$

$$= \mathbb{E}(X_1 + \cdots + X_{n-1}) + \mathbb{E}(X_n) = \cdots$$

Coin flipping again

$$E(X) = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

Suppose we flip a coin independently n times with probability p of coming up Heads each time. Let the r.v. Z be the number of Heads in the n coin flips. What is the $\mathbb{E}(Z)$?

Let $X_i = \begin{cases} 1 & \text{if } i^{th} \text{ coin toss comes up H's} \\ 0 & \text{o.w.} \end{cases}$

$$Z = X_1 + X_2 + X_3 + \cdots + X_n$$

LOE

$$E(Z) = E(X_1 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$E(X_i) = 1 \cdot Pr(i^{th} \text{ toss H}) + 0 \cdot Pr(i^{th} \text{ toss T})$$

$$= Pr(i^{th} \text{ toss is H}) = p$$

$$E(Z) = \underbrace{E(X_1)}_{p} + \cdots + \underbrace{E(X_n)}_{p} = np$$

Example – Coin Tosses

we flip n coins, each one heads with probability p

Z is the number of heads, what is $\mathbb{E}(Z)$?

- $X_i = \begin{cases} 1, & i\text{–th coin–flip is heads} \\ 0, & i\text{–th coin–flip is tails.} \end{cases}$

Fact. $Z = X_1 + \cdots + X_n$

Linearity of Expectation:

$$\mathbb{E}(Z) = \mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) \quad = n \cdot p$$

$\mathbb{P}(X_i = 1) = p$
$\mathbb{P}(X_i = 0) = 1 - p$

$$\mathbb{E}(X_i) = p \cdot 1 + (1 - p) \cdot 0 = p$$

Computing complicated expectations

Often boils down to the following three steps

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Observe linearity of expectation.

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n).$$

- Conquer: Compute the expectation of each X_i

Often, X_i are indicator (0/1) random variables.

X_i:

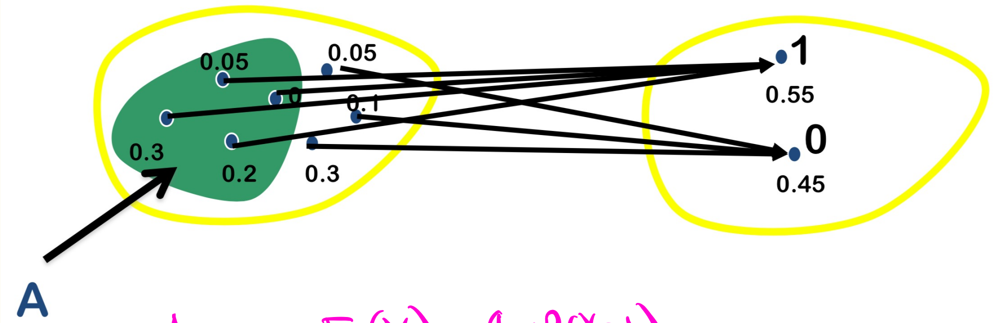
Indicator random variable

For any event A, can define the indicator random variable X for A

$$X = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{if event A does not occur} \end{cases}$$

$\mathbb{P}(X = 1) = \mathbb{P}(A) = 0.55$

$\mathbb{P}(X = 0) = 1 - \mathbb{P}(A)$



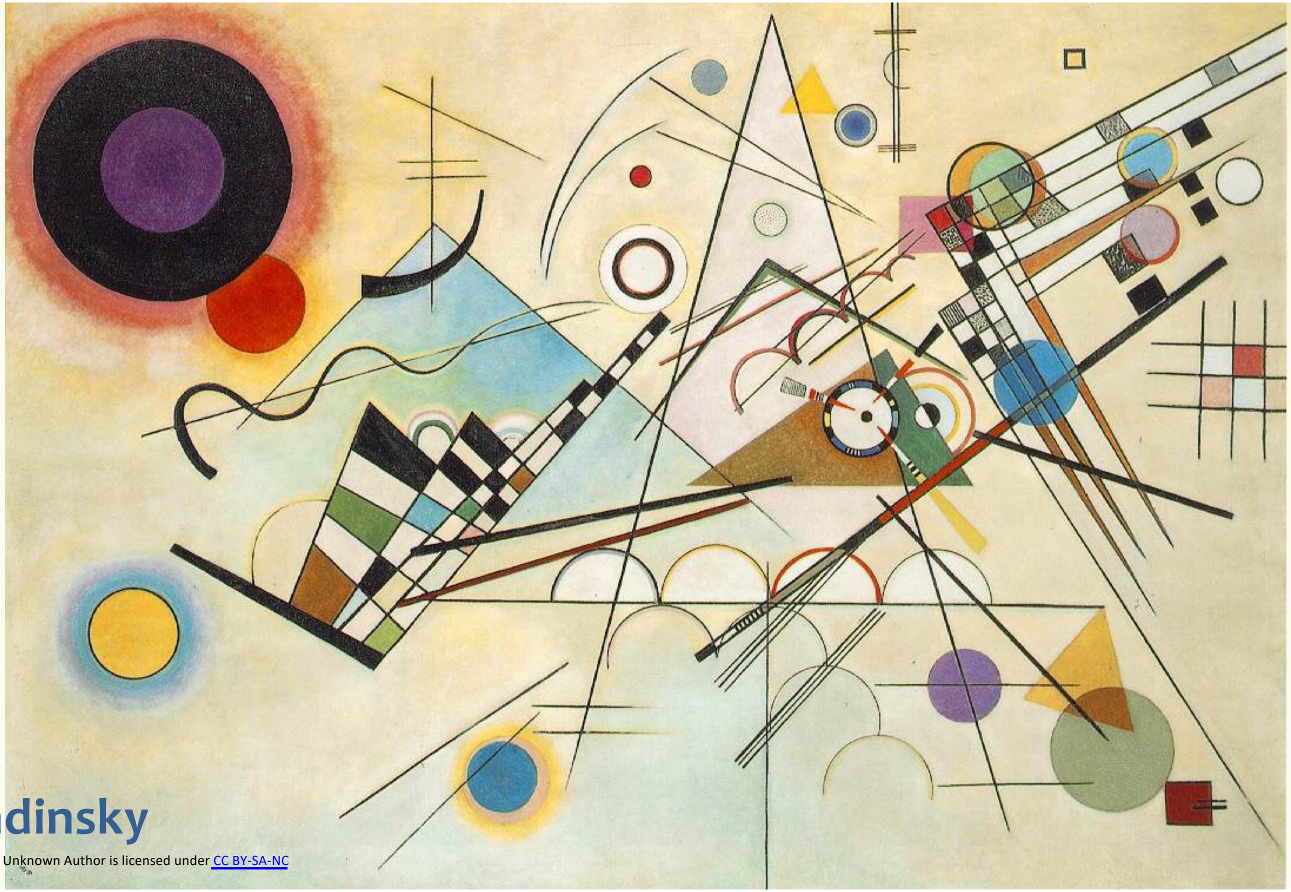
X indicator r.v. A

$E(X) = 1 \cdot P(X=1) + 0 \cdot P(X=0)$

$= P(X=1) = P(A)$

X_i indicator if i^{th} toss H's.

$E(X_i) = Pr(i^{th}$ toss H's$) = p$



Kandinsky

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW
- what is $\mathbb{E}(X)$?

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW
- what is $\mathbb{E}(X)$?
- Use Linearity of Expectation

Decompose: What is X_i?

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

~~$E(X \cdot Y) = E(X)E(Y)$~~

Example: Returning Homeworks

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1/6	1, 2, 3	3
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1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

Decompose: X_i indicates if student i got their own HW back

LOE:

Conquer: What is $\mathbb{E}(X_i)$? A. $\frac{1}{n}$ B. $\frac{1}{n-1}$ C. ½

https://pollev.com/ annakarlin185

Pairs with same birthday

X: # pairs of students that have same bday

- In a class of m students, on average how many pairs of people have the same birthday?

Decompose:

$$X = \sum_{(i,j) \text{ unordered pairs}} X_{ij}$$

LOE:

$$E(X) = \sum_{(i,j) \text{ unordered}} E(X_{ij}) = \binom{m}{2}\frac{1}{365}$$

Conquer:

$$X_{ij} = \begin{cases} 1 & \text{student } i \text{ \& student } j \text{ have same bday} \\ 0 & \text{o.w.} \end{cases}$$

$$E(X_{ij}) = \frac{1}{365}$$

$$= \sum_{bday=1}^{365} \frac{1}{365}\cdot\frac{1}{365} = \frac{1}{365}$$

$X_{12} + X_{13} + X_{23}$

Linearity of Expectation – Even stronger

Theorem. For any random variables $X_1, \dots, X_n$, and real numbers $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}(a_1X_1 + \cdots + a_nX_n) = a_1\mathbb{E}(X_1) + \cdots + a_n\mathbb{E}(X_n).$$

$$E(5X_1 + 7X_2 + 10) = 5E(X_1) + 7E(X_2) + 10$$

Very important: In general, we do <u>not</u> have $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$