Lecture 8: Random Variables and Expectation
Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation
Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

- What is the total of two dice rolls?
- What is the number of coin tosses needed to see the first head?
- What is the number of heads among 2 coin tosses?

\[\begin{align*}
\text{HT} & \rightarrow 1 \\
\text{TT, ZZ} & \rightarrow 3 \\
\text{TTT, TTH, THH} & \rightarrow 5
\end{align*}\]
Random Variables

Definition. A random variable (RV) for a probability space $(\Omega, \mathcal{P})$ is a function $X : \Omega \rightarrow \mathbb{R}$.

The set of values that $X$ can take on is called its range/support $\Omega_X$.

Example. Number of heads in 2 independent coin flips $\Omega = \{HH, HT, TH, TT\}$.
RV Example

20 balls labeled 1, 2, …, 20 in a bin

– Draw a subset of 3 uniformly at random

– Let $X = \text{maximum of the 3 numbers on the balls}$

  • Example: $X(2, 7, 5) = 7$
  • Example: $X(15, 3, 8) = 15$

– What is $|\Omega_X|$?

\[\{1, 2, 3\}\]
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Random variables partition the sample space.

\[ \sum_{x \in X(\Omega)} P(X = x) = 1 \]
**Definition.** For a RV $X: \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} \overset{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $\mathbb{P}(X = x) = \mathbb{P}(\{X = x\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$ where $\mathbb{P}(X = x)$ is the probability mass function (PMF) of $X$

Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$
**Probability Mass Function (PMF)**

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Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$

You also see this notation (e.g. in book):

$$\mathbb{P}(X = x) = p_{X}(x)$$
Probability Mass Function

Flipping two independent coins

\[ X = \text{number of heads in the two flips} \]

\[ X(\text{HH}) = 2 \quad X(\text{HT}) = 1 \quad X(\text{TH}) = 1 \quad X(\text{TT}) = 0 \]

\[ \Omega = \{\text{HH, HT, TH, TT}\} \]

\[ \Omega_x = \{0, 1, 2\} \]

What is \( Pr(X = k) \)?

\[ Pr(X = k) = \begin{cases} 
\frac{1}{4} & k = 0 \\
\frac{1}{2} & k = 1 \\
\frac{1}{4} & k = 2 \\
0 & \text{otherwise} 
\end{cases} \]

\[ Pr(X = 2) \to \frac{1}{4} \]

\[ Pr(X = 0, 5) \]
RV Example

20 balls labeled 1, 2, …, 20 in a bin
- Draw a subset of 3 uniformly at random
- Let $X =$ maximum of the 3 numbers on the balls

What is $P(X = 20)$?

$$P(X = 20) = \frac{P(\varepsilon \ldots \varepsilon)}{\Omega} = \frac{\binom{19}{2}}{\binom{20}{3}}$$

Poll:
https://pollev.com/annakarlin85

A. \(\frac{\binom{20}{2}}{\binom{20}{3}}\)
B. \(\frac{\binom{19}{2}}{\binom{20}{3}}\)
C. \(19^2 / \binom{20}{3}\)
D. \(19 \cdot 18 / \binom{20}{3}\)
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Cumulative Distribution Function (CDF)

**Definition.** For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of where $X$ specifies for any real number $x$, the probability that $X \leq x$.

$$F_X(x) = \Pr(X \leq x)$$

Go back to 2 coin clips, where $X$ is the number of heads

$$\Pr(X = x) = \begin{cases} 
\frac{1}{4}, & x = 0 \\
\frac{1}{2}, & x = 1 \\
\frac{1}{4}, & x = 2 
\end{cases}$$
Cumulative Distribution Function (CDF)

**Definition.** For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of where $X$ specifies for any real number $x$, the probability that $X \leq x$.

$$F_X(x) = \Pr(X \leq x)$$

Go back to 2 coin clips, where $X$ is the number of heads

$$\Pr(X = x) = \begin{cases} 
\frac{1}{4}, & x = 0 \\
\frac{1}{2}, & x = 1 \\
\frac{1}{4}, & x = 2 
\end{cases}$$

$$F_X(x) = \begin{cases} 
0, & x < 0 \\
\frac{1}{4}, & 0 \leq x < 1 \\
\frac{3}{4}, & 1 \leq x < 2 \\
1, & 2 \leq x 
\end{cases}$$
Example: Returning Homeworks

• Class with 3 students, randomly hand back homeworks. All permutations equally likely.
• Let $X$ be the number of students who get their own HW

<table>
<thead>
<tr>
<th>Pr($\omega$)</th>
<th>$\omega$</th>
<th>$X(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/6</td>
<td>1, 2, 3</td>
<td>3</td>
</tr>
<tr>
<td>1/6</td>
<td>1, 3, 2</td>
<td>1</td>
</tr>
<tr>
<td>1/6</td>
<td>2, 1, 3</td>
<td>1</td>
</tr>
<tr>
<td>1/6</td>
<td>2, 3, 1</td>
<td>0</td>
</tr>
<tr>
<td>1/6</td>
<td>3, 1, 2</td>
<td>0</td>
</tr>
<tr>
<td>1/6</td>
<td>3, 2, 1</td>
<td>1</td>
</tr>
</tbody>
</table>

$\Omega_X = \{3, 3, 1, 0, 3\}$

$P(X = x) = \begin{cases} 
\frac{1}{6} & x = 3 \\
\frac{1}{2} & x = 1 \\
\frac{1}{3} & x = 0 \\
0 & \text{otherwise}
\end{cases}$

$P(X \leq x)$

$P(X \leq 10)$
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Expectation (Idea)

What is the *expected* number of heads in 2 independent flips of a fair coin?
**Definition.** Given a discrete RV $X: \Omega \rightarrow \mathbb{R}$, the expectation or expected value of $X$ is

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$$

or equivalently

$$E[X] = \sum_{x \in X(\Omega)} x \cdot \Pr(X = x)$$

Intuition: “Weighted average” of the possible outcomes (weighted by probability)
Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

\[
\mathbb{E}[X] = \sum_{x \in \Omega} x \cdot P(X = x)
\]

\[
= 3 \cdot P(X = 3) + 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{3}
\]

\[
= \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)
\]

\[
= 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}
\]

\[
= 1
\]
Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability $p$ of being heads. Keep flipping independent flips until heads. Let $X$ be the number of flips until heads.

What is: $\Pr(X = 1) = p$

What is: $\Pr(X = 2) = (1-p)p$

What is: $\Pr(X = 3) = (1-p)^2p$

What is: $\Pr(X = k) = (1-p)^{k-1}p$
Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability $p$ of being heads. Keep flipping independent flips until heads. Let $X$ be the number of flips until heads. What is $E[X]$?

$$E[X] = \frac{1}{p}$$

$$E[X] = \sum_{k \in \mathbb{N}} k \cdot P(X = k) = \sum_{k=1}^{\infty} k \cdot P(X = k)$$

$$= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = \frac{1}{p^2} \cdot p = \frac{1}{p} = E[X]$$

Geometric Series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \forall \delta x \text{ both sides}$$

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Sub $x = (1-p)$
Students on a bus

A group of 120 students are driven on 3 buses to a football game. There are 36 students in the first bus, 40 in the second and 44 in the third. Let $Y$ be the number of students on a uniformly random bus. What is the pmf of $Y$ and $E(Y)$? When the buses arrive, one of the 120 students is randomly chosen. Let $X$ denote the number of students on the bus of the randomly chosen student. What is the pmf of $X$ and what is $E(X)$?
Coin flipping again

Suppose we flip a coin with probability $p$ of coming up Heads $n$ times. Let $X$ be the number of Heads in the $n$ coin flips. What is the pmf of $X$?