Lecture 4: Intro to discrete probability
Probability

• We want to model uncertainty.
  – i.e., outcome not determined a-priori
  – E.g. throwing dice, flipping a coin...
  – We want to numerically measure likelihood of outcomes = probability.
  – We want to make complex statements about these likelihoods.
• We will not argue why a certain physical process realizes the probabilistic model we study
  – Why is the outcome of the coin flip really “random”?
• First part of class: “Discrete” probability theory
  – Experiment with finite / discrete set of outcomes.
  – Will explore countably infinite and continuous outcomes later
Agenda

• Events
• Probability
• Equally Likely Outcomes
• Probability Axioms and Beyond Equally Likely Outcomes
• More Examples
Definition. A sample space $\Omega$ is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
Events

**Definition.** An event $E \subseteq \Omega$ is a subset of possible outcomes.

**Examples:**

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die: $E = \{2, 4, 6\}$
**Events**

**Definition.** An event $E \subseteq \Omega$ is a subset of possible outcomes.

**Examples:**
- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die: $E = \{2, 4, 6\}$

**Definition.** Events $E$ and $F$ are **mutually exclusive** if $E \cap F = \emptyset$ (i.e., can’t happen at same time)

**Examples:**
- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$
**Example: 4-sided Dice**

Suppose I roll two 4-sided dice. Let $D_1$ be the value of the blue die and $D_2$ be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$

B. $D_1 + D_2 = 6$

C. $D_1 = 2 \times D_2$

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Example: 4-sided Dice

Suppose I roll two 4-sided dice. Let $D_1$ be the value of the blue die and $D_2$ be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$
   
   $A = \{(1,1), (1,2), (1,3), (1,4)\}$

B. $D_1 + D_2 = 6$
   
   $B = \{(2,4), (3,3), (4,2)\}$

C. $D_1 = 2 \times D_2$
   
   $C = \{(2, 1), (4, 2)\}$

\[\begin{array}{cccc}
&D1 & & \\
\hline
1 & (1, 1) & (1, 2) & (1, 3) & (1, 4) \\
2 & (2, 1) & (2, 2) & (2, 3) & (2, 4) \\
3 & (3, 1) & (3, 2) & (3, 3) & (3, 4) \\
4 & (4, 1) & (4, 2) & (4, 3) & (4, 4) \\
\end{array}\]
Example: 4-sided Dice, Mutual Exclusivity

Are $A$ and $B$ mutually exclusive?
How about $B$ and $C$?
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A. $D_1 = 1$

B. $D_1 + D_2 = 6$

C. $D_1 = 2 \times D_2$

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<th>A &amp; B</th>
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<td>(a) Yes</td>
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<td>(b) Yes</td>
<td>No</td>
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<td>(c) No</td>
<td>Yes</td>
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<td>(d) No</td>
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Die 1 ($D_1$)

Die 2 ($D_2$)
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Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

$$\mathbb{P}: \Omega \rightarrow [0, 1]$$

that maps outcomes $$\omega \in \Omega$$ to probabilities.

– Also use notation: $$\mathbb{P}(\omega) = P(\omega) = \Pr(\omega)$$
Example – Coin Tossing

Imagine we toss one coin – outcome can be heads or tails.

\[ \Omega = \{H, T\} \]

depends! What do we want to model?!

Fair coin toss

\[ \mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2} = 0.5 \]
Example – Coin Tossing

Imagine we toss one coin – outcome can be heads or tails.

$$\Omega = \{H, T\}$$

$$\mathbb{P}$$? Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(H) = 0.85, \quad \mathbb{P}(T) = 0.15$$
Definition. A (discrete) probability space is a pair \((\Omega, \mathbb{P})\) where:

- \(\Omega\) is a set called the **sample space**.
- \(\mathbb{P}\) is the **probability measure**, a function \(\mathbb{P}: \Omega \to [0,1]\) such that:
  - \(\mathbb{P}(\omega) \geq 0\) for all \(\omega \in \Omega\)
  - \(\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1\)
**Probability space**

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Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Either finite or infinite countable (e.g., integers)

Set of possible elementary outcomes

Specify Likelihood (or probability) of each elementary outcome
**Uniform Probability Space**

**Definition.** A **uniform probability space** is a pair $(\Omega, \mathbb{P})$ such that

$$\mathbb{P}(x) = \frac{1}{|\Omega|}$$

for all $x \in \Omega$.

**Examples:**

- Fair coin $P(x) = \frac{1}{2}$
- Fair 6-sided die $P(x) = \frac{1}{6}$

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = \sum_{\omega \in \Omega} \frac{1}{|\Omega|} = 1$$
Events

**Definition.** An event in a probability space \((\Omega, \mathbb{P})\) is a subset \(\mathcal{A} \subseteq \Omega\). Its probability is

\[ \mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega) \]

Convenient abuse of notation: \(\mathbb{P}\) is extended to be defined over sets. \(\mathbb{P}(\omega) = \mathbb{P}({\omega})\)
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Think back to 4-sided die. Suppose each die is fair. What is the probability of event \( B \)? \( \Pr(B) = ? \? ? \)

**B. \( D_1 + D_2 = 6 \)** \[ B = \{(2,4), (3,3)(4,2)\} \]

\[
\Pr(\text{B}_1 + \text{B}_2 = 6) = \frac{3}{16}
\]
Equally Likely Outcomes

If \((\Omega, P)\) is a **uniform** probability space, then for any event \(E \subseteq \Omega\), then

\[
P(E) = \frac{|E|}{|\Omega|}
\]

This follows from the definitions of the prob. of an event and uniform probability spaces.

\[
\text{Pr}(E) = \sum_{\omega \in E} \text{Pr}(\omega) = \sum_{\omega \in E} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}
\]
Example – Coin Tossing

Toss a coin 100 times. Each outcome is equally likely. What is the probability of seeing 50 heads?

\[ \Pr(E) = \frac{1E!}{121} \]

\[ |E| = 2^{100} \]

\[ |E| = \binom{100}{50} \]

(A) \( \frac{1}{2} \)

(B) \( \frac{1}{250} \)

(C) \( \frac{\binom{100}{50}}{2^{100}} \)

(D) Not sure

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Brain Break
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Axioms of Probability

Let $\Omega$ denote the sample space and $E, F \subseteq \Omega$ be events. Note this is applies to any probability space (not just uniform).

Axiom 1 (Non-negativity): $P(E) \geq 0$.
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If $E$ and $F$ are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
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- \(\Omega\) is a set called the **sample space**.
- \(\mathbb{P}\) is the **probability measure**, a function \(\mathbb{P}: \Omega \rightarrow [0,1]\) such that:
  - \(\mathbb{P}(\omega) \geq 0\) for all \(\omega \in \Omega\)
  - \(\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1\)

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.
Non-equally Likely Outcomes

Probability spaces can have non-equally likely outcomes.
More Examples of Non-equally Likely Outcomes
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Suppose I had a two, fair, 6-sided dice that we roll once each. What is the probability that we see **at least one 3 in the two rolls**.

\[ \mathcal{E} = \{(1,1), (1,2), \ldots, (6,6)\} \]

\[ |\mathcal{E}| = 36. \]

E: outcomes that include at least one 3

\[ \Pr(E) = 1 - \Pr(E^c) \]

\[ = 1 - \frac{|E^c|}{|\mathcal{E}|} = 1 - \frac{25}{36} = \frac{11}{36} \]
Example: Birthday “Paradox”

Suppose we have a collection of \( n \) people in a room. What is the probability that at least 2 people share a birthday? Assuming there are 365 possible birthdays, with uniform probability for each day.

\[
E = E^c = \Omega \setminus E
\]

\[
\mathbb{P}(E) = 1 - \mathbb{P}(E^c) = 1 - \frac{1}{365^n}
\]

\[
|E| = 365 \cdot 364 \cdots 365-n+1 = \frac{365!}{(365-n)!} = \binom{365}{n}
\]
Example: Birthday “Paradox” cont.

\[ 1 - \frac{P(365, n)}{365^n} \]

\[ n = 23 \quad \text{> 0.5} \]
\[ n = 60 \quad \text{> 0.98} \]

May 8

Pr(group of people I screen who has bday)

\[ = 1 - \text{Pr}(\text{nobody has bday}) = 1 - \left( \frac{364^n}{365^n} \right) \]

n = 23

64

0.06

64

0.16

150

0.23
Example: Birthday “Paradox” cont.