CSE 312

Foundations of Computing II

Lecture 4: Intro to discrete probability

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OF COMPUTER SCIENCE & ENGINEERING

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺
Plus few slides from Berkeley CS 70

Probability

- We want to model uncertainty.
  - i.e., outcome not determined a-priori
  - E.g. throwing dice, flipping a coin…
  - We want to numerically measure likelihood of outcomes = probability.
  - We want to make complex statements about these likelihoods.
- We will not argue <u>why</u> a certain physical process realizes the probabilistic model we study
  - Why is the outcome of the coin flip really "random"?
- First part of class: "Discrete" probability theory
  - Experiment with finite / discrete set of outcomes.
  - Will explore countably infinite and continuous outcomes later

Agenda

- Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Sample Space

\Omega

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

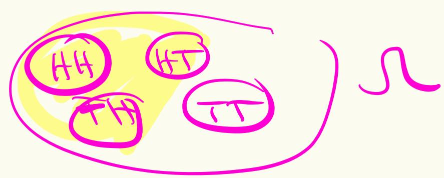
- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

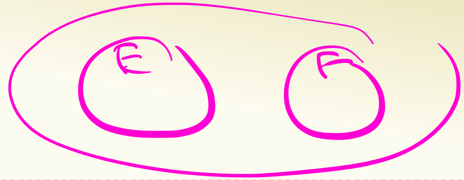
Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$



Events

Ω

Ω 

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:
- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Definition. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$ (i.e., can't happen at same time)

Examples:
- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$

Example: 4-sided Dice

$|S| = 16$

Suppose I roll two 4-sided dice Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D1 = 1$

B. $D1 + D2 = 6$

C. $D1 = 2 * D2$

Die 2 (D2)

Die 1 (D1)	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice

Suppose I roll two 4-sided dice Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. D1 = 1

$A = \{(1,1),(1,2),(1,3),(1,4)\}$

B. D1 + D2 = 6

$B = \{(2,4),(3,3),(4,2)\}$

C. D1 = 2 * D2

$C = \{(2,1),(4,2)\}$

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice, Mutual Exclusivity

no overlap.

Are A and B mutually exclusive?
How about B and C?
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	A & B	B & C
(a)	Yes	Yes
(b)	Yes	No
(c)	No	Yes
(d)	No	No

A. D1 = 1

B. D1 + D2 = 6

C. D1 = 2 * D2

Die 2 (D2)

Die 1 (D1)

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1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
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Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

$P(\omega)$

$\omega \in \Omega$

Will define a function

$$\mathbb{P}\colon\ \Omega \to [0,1]$$

that maps outcomes $\omega \in \Omega$ to probabilities.

– Also use notation: $\mathbb{P}(\omega) = P(\omega) = \Pr(\omega)$

Example – Coin Tossing

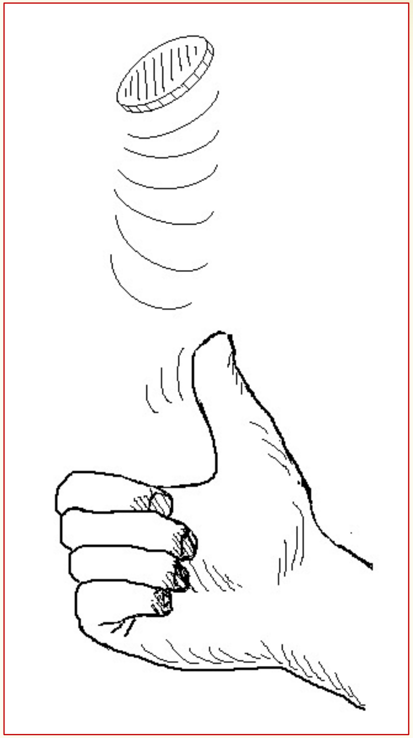
Imagine we toss one coin – outcome can be **heads** or **tails.**

$\Omega = \{\mathrm{H}, \mathrm{T}\}$

$\mathbb{P}$? Depends! What do we want to model?!

Fair coin toss

$$\mathbb{P}(\mathrm{H}) = \mathbb{P}(\mathrm{T}) = \frac{1}{2} = 0.5$$



Example – Coin Tossing

Imagine we toss one coin – outcome can be **heads** or **tails.**

$$\Omega = \{\mathrm{H}, \mathrm{T}\}$$

$\mathbb{P}$? Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(\mathrm{H}) = 0.85, \qquad \mathbb{P}(\mathrm{T}) = 0.15$$

Probability space

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

- Ω is a set called the **sample space**.
- $\mathbb{P}$ is the **probability measure**, a function $\mathbb{P}: \Omega \to [0,1]$ such that:
  - $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

set of possible outcomes

every outcome

some outcome occurs in Ω

Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

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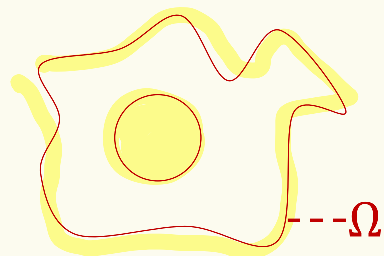
Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Uniform Probability Space

Definition. A **uniform probability space** is a pair $(\Omega, \mathbb{P})$ such that

$$\mathbb{P}(x) = \frac{1}{|\Omega|}$$

for all $x \in \Omega$.

$\sum_{\omega \in \Omega} Pr(\omega) = \sum_{\omega \in \Omega} \frac{1}{|\Omega|}$

$= 1$

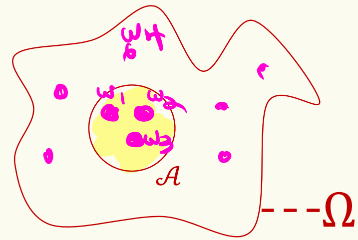
Examples:

- Fair coin $P(x) = \frac{1}{2}$
- Fair 6-sided die $P(x) = \frac{1}{6}$

Events

Definition. An **event** in a probability space $(\Omega, \mathbb{P})$ is a subset $\mathcal{A} \subseteq \Omega$. Its probability is

$$\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)$$



$Pr(\mathcal{A}) = Pr(\omega_1) + Pr(\omega_2) + Pr(\omega_3)$

Convenient abuse of notation: $\mathbb{P}$ is extended to be defined over **sets**. $\mathbb{P}(\omega) = \mathbb{P}(\{\omega\})$

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Example: 4-sided Dice, Event Probability

Think back to 4-sided die. Suppose each die is fair. What is the probability of event B? $\Pr(B) = ???$

B. D1 + D2 = 6

$B = \{(2,4), (3,3)(4,2)\}$

$\Pr(\omega) = \frac{1}{16}$

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	$\boxed{(2, 4)}$
3	(3, 1)	(3, 2)	$\boxed{(3, 3)}$	(3, 4)
4	(4, 1)	$\boxed{(4, 2)}$	(4, 3)	(4, 4)

$\Pr(D_1 + D_2 = 6)$

$= \underbrace{\Pr((4,2))}_{\frac{1}{16}} + \underbrace{\Pr((3,3))}_{\frac{1}{16}} + \underbrace{\Pr((2,4))}_{\frac{1}{16}} = \frac{3}{16}$

Equally Likely Outcomes

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$, then

$$P(E) = \frac{|E|}{|\Omega|}$$

This follows from the definitions of the prob. of an event and uniform probability spaces.

$$Pr(E) = \sum_{\omega \in E} Pr(\omega) = \sum_{\omega \in E} \boxed{\frac{1}{|\Omega|}} = \frac{|E|}{|\Omega|}$$

Example – Coin Tossing

$\Omega = \{$all sequences of H's & T's of length 100$\}$

$E = \{$all sequences of H's & T's of length 100 with exactly 50 H's$\}$

Toss a coin 100 times. Each outcome is **equally likely**. What is the probability of seeing 50 heads?

E

$$Pr(E) = \frac{|E|}{|\Omega|}$$

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(A) $\frac{1}{2}$

(B) $\frac{1}{2^{50}}$

(C) $\boxed{\frac{\binom{100}{50}}{2^{100}}}$

(D) Not sure

$|\Omega| = 2^{100}$

H/T H H H — — — —

1 100

$|E| = \binom{100}{50}$

Brain Break

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Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is applies to **any** probability space (not just uniform)

Axiom 1 (Non-negativity): $P(E) \geq 0.$

Axiom 2 (Normalization): $P(\Omega) = 1$

Axiom 3 (Countable Additivity): If *E* and *F* are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E).$

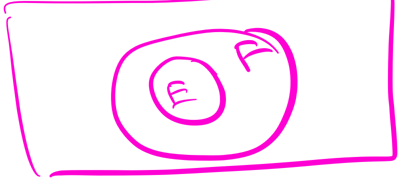
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$

Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$E \cup E^c = \Omega$

$\Pr(E \cup E^c) = \boxed{\Pr(E) + \Pr(E^c)}$

$\Pr(\Omega) = 1$



Review Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

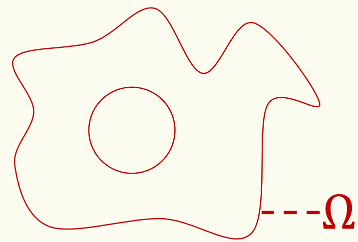
- Ω is a set called the **sample space.**
- $\mathbb{P}$ is the **probability measure,**

  a function $\mathbb{P}: \Omega \to [0,1]$ such that:

  – $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$

  – $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Set of possible **elementary outcomes**



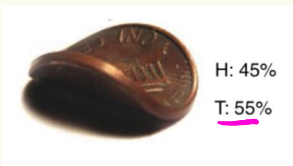
Specify Likelihood (or probability) of each **elementary outcome**

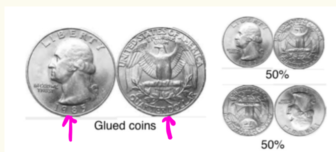
Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Non-equally Likely Outcomes

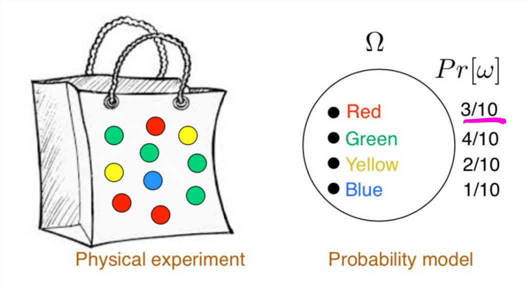
Probability spaces can have **non-equally likely outcomes.**

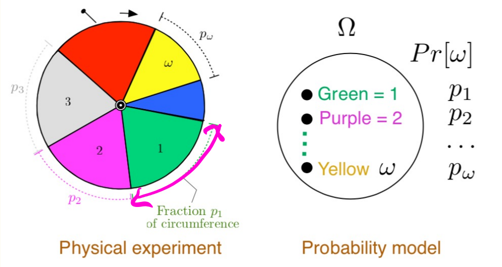




{HH, HT, TH, TT}
0 0.5 0.5 0

More Examples of Non-equally Likely Outcomes





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Example: Dice Rolls

uniform prob space.

Suppose I had a two, fair, 6-sided dice that we roll once each. What is the probability that we see *at least one 3 in the two rolls*.

$$\Omega = \{(1,1), (1,2) \ldots (6,6)\}$$

$$|\Omega| = 36.$$

E : outcomes that include at least one 3

D1	D2
1	1
2	2
4	4
5	5
6	6

$$Pr(E) = 1 - Pr(\underset{\text{no } 3}{E^c})$$

$$= 1 - \frac{|E^c|}{|\Omega|} = 1 - \frac{25}{36} = \frac{11}{36}$$

$\overline{E} = E^c = \Omega \setminus E$

Example: Birthday "Paradox"

uniform prob space.

Suppose we have a collection of n people in a room. What is the probability that at least 2 people share a birthday? Assuming there are 365 possible birthdays, with uniform probability for each day.

$p_1 \quad p_2 \quad \ldots \quad p_n$

Mar 18 Sept 2

(Sept 1, Sept 1)
(Sept 1, Sept 2)
(Sept 1, Oct 2)

$\Omega = \{(b_1, b_2, \ldots, b_n) \mid 1 \le b_i \le 365\}$

$|\Omega| = 365^n$

E: There are 2 people w/ same bday.

$Pr(E) = 1 - Pr(\overline{E}) = 1 - \frac{|\overline{E}|}{365^n}$

no 2 people have same bday

$|\overline{E}| = 365 \cdot 364 \cdots 365-n+1 = \frac{365!}{(365-n)!} = P(365, n)$

Example: Birthday "Paradox" cont.

$$1 - \frac{P(365, n)}{365^n}$$

$n = 23$ > 0.5

$n = 60$ > 0.98

May 8

Pr(group of n people ∃ someone who has bday May 8)

$$= 1 - \underline{Pr\left(\text{nobody has May 8}\right)} = 1 - \frac{364^n}{365^n}$$

$n = 23$ 0.06

64 0.16

150 0.23

Example: Birthday “Paradox” cont.