Lecture 3: Pigeonhole principle + practice with counting

Slide Credit: Based on Stefano Tessaro’s slides for 312 19au incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊
Recap (1)

**Product Rule:** In a sequential process, there are
- $n_1$ choices for the first step,
- $n_2$ choices for the second step (given the first choice), \ldots, and
- $n_m$ choices for the $m$\textsuperscript{th} step (given the previous choices),
then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$

**Application.** # of $k$-element sequences of distinct symbols (a.k.a. $k$-permutations) from $n$-element set is

$$P(n, k) = n \times (n - 1) \times \cdots \times (n - k + 1) = \frac{n!}{(n - k)!}$$
**Recap (2)**

**Combination:** If order does not matter, then count the number of ordered objects, and then divide by the number of orderings.

**Applications.** The number of subsets of size \( k \) of a set of size \( n \) is

\[
\binom{n}{k} = \frac{n!}{k! (n-k)!}
\]

*Binomial coefficient* (verbalized as “\( n \) choose \( k \)”)
Agenda

- Pigeonhole Principle
- More practice with counting
Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes
Pigeonhole Principle: Idea

If 11 children have to share 3 cakes, at least one cake must be shared by how many children?
Pigeonhole Principle – More generally

If there are \( n \) pigeons in \( k < n \) holes, then one hole must contain at least \( \frac{n}{k} \) pigeons!

**Proof.** Assume there are \( < \frac{n}{k} \) pigeons per hole. Then, there are \( < k \frac{n}{k} = n \) pigeons overall. Contradiction!
Pigeonhole Principle – Better version

If there are $n$ pigeons in $k < n$ holes, then one hole must contain at least $\left\lfloor \frac{n}{k} \right\rfloor$ pigeons!
Pigeonhole Principle – Better version

If there are \( n \) pigeons in \( k < n \) holes, then one hole must contain at least \( \left\lceil \frac{n}{k} \right\rceil \) pigeons!

Reason. Can’t have fractional number of pigeons

Syntax reminder:
• Ceiling: \([x]\) is \(x\) rounded up to the nearest integer (e.g., \([2.731]\) = 3)
• Floor: \([x]\) is \(x\) rounded down to the nearest integer (e.g., \([2.731]\) = 2)
In a room with 367 people, there are at least two with the same birthday.

Solution:
1. 367 pigeons = people
2. 365 holes = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday
Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps:

1. Identify pigeons
2. Identify pigeonholes
3. Specify a rule for assigning pigeons to pigeonholes
4. Apply PHP
Pigeonhole Principle – Example (Surprising?)

In every set $S$ of 100 integers, there are at least two elements whose difference is a multiple of 37.

When solving a PHP problem:
1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP
Agenda

- Pigeonhole Principle
- More practice with counting
Quick Review of Cards

- 52 total cards
- 13 different ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

How many possible 5 card hands?
A **straight** is five consecutive rank cards of any suit. How many possible straights?
Counting Cards II

A flush is a five card hand all of the same suit.
How many possible flushes?
Counting Cards III

A **flush** is five card hand all of the same suit. How many possible flushes?

\[ 4 \cdot \binom{13}{5} = 5148 \]

How many flushes are **NOT** straights?
A flush is a five card hand all of the same suit. How many possible flushes?

\[ 4 \cdot \binom{13}{5} = 5148 \]

How many flushes are NOT straights?

\[ = \text{#flush} - \text{#flush and straight} \]

\[ (4 \cdot \binom{13}{5} = 5148) - 10 \cdot 4 \]
For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

- No sequence ➔ under counting
- Many sequences ➔ over counting
For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

**Sleuth’s Criterion (Rudich)**

No sequence ➔ under counting  
Many sequences ➔ over counting

**EXAMPLE:** How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.  \( \binom{4}{3} \cdot \binom{49}{2} \)
For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

No sequence $\Rightarrow$ under counting  Many sequences $\Rightarrow$ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.  \[
\binom{4}{3} \cdot \binom{49}{2}
\]

Poll:
A. Correct
B. Overcount
C. Undercount

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Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting  Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards. \( \binom{4}{3} \cdot \binom{49}{2} \)
For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

When in doubt, break up into disjoint sets you know how to count, and then use the sum rule.
For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

\[ = \# \text{ 5 card hand containing exactly 3 Aces} + \# \text{ 5 card hand containing exactly 4 Aces} \]

\[ = \binom{4}{3} \cdot \binom{48}{2} + \binom{48}{1} \]
8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column?

Poll:
A. $\binom{64}{3}$
B. $\binom{8}{3} \cdot \binom{8}{3}$
C. $8^2 \cdot 7^2 \cdot 6^2$
D. I don’t know.

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8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column?

Sequential process:
1. Column for pawn
2. Row for pawn
3. Column for bishop
4. Row for bishop
5. Column for knight
6. Row for knight

\[(8 \cdot 7 \cdot 6)^2\]
Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don’t share a row or a column

Poll:
A. $8^2 \cdot 7^2$
B. $\binom{8}{2} \cdot \binom{8}{2}$
C. $\frac{8^2 \cdot 7^2}{2}$
D. I don’t know.

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Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don’t share a row or a column

Pretend Rooks are different
1. Column for rook1
2. Row for rook1
3. Column for rook2
4. Row for rook2

Remove the order between two rooks

\[(8 \cdot 7)^2 / 2\]
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, Plain

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

Product Rule: In a sequential process, there are
• $n_1$ choices for the first step,
• $n_2$ choices for the second step (given the first choice), ..., and
• $n_m$ choices for the $m^{th}$ step (given the previous choices),
then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, Plain

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

Try 1: First choose # of Chocolate, then # Lemon, then # Maple, then # Glazed, then # Plain

Product Rule: In a sequential process, there are
• $n_1$ choices for the first step,
• $n_2$ choices for the second step (given the first choice), ..., and
• $n_m$ choices for the $m^{th}$ step (given the previous choices),
then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, Plain

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

Try 2: First choose type of donut 1, then type of donut 2,....., then type of donut 12.

Product Rule: In a sequential process, there are
• $n_1$ choices for the first step,
• $n_2$ choices for the second step (given the first choice), ..., and
• $n_m$ choices for the $m^{th}$ step (given the previous choices),
then the total number of outcomes is $n_1 \times n_2 \times ... \times n_m$
Bijection Rule

If there is a bijection (one-to-one and onto mapping) between set A and set B, then $|A| = |B|$.

There is a bijection between sequences of length 16 with 4 bars and 12 stars and the doughnut choices.
You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, plain.

How many ways are there to choose a dozen donuts when donuts of the same type are indistinguishable?

There is a bijection between sequences of length 16 with 4 bars and 12 stars and the donut choices.
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, plain
How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, plain

How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?

Mental process:
1. Place one donut in each flavor bin
2. Choose the remaining 7 donuts without restriction

\[
\binom{7 + 5 - 1}{5 - 1}
\]
Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Stars and bars
Counting is **NOT** for kindergarteners