Lecture 3: Pigeonhole principle + practice with counting
Recap (1)

**Product Rule:** In a sequential process, there are
- \( n_1 \) choices for the first step,
- \( n_2 \) choices for the second step (given the first choice), ..., and
- \( n_m \) choices for the \( m \)th step (given the previous choices),
then the total number of outcomes is \( n_1 \times n_2 \times \cdots \times n_m \)

**Application.** # of \( k \)-element sequences of distinct symbols
(a.k.a. \( k \)-permutations) from \( n \)-element set is

\[
P(n, k) = n \times (n - 1) \times \cdots \times (n - k + 1) = \frac{n!}{(n - k)!}
\]
Recap (2)

**Combination:** If order does not matter, then count the number of ordered objects, and then divide by the number of orderings.

**Applications.** The number of subsets of size $k$ of a set of size $n$ is

$$ \binom{n}{k} = \frac{n!}{k!(n-k)!} $$

*Binomial coefficient* (verbalized as “$n$ choose $k$”)
Agenda

• Pigeonhole Principle
• More practice with counting
Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes
If 11 children have to share 3 cakes, at least one cake must be shared by how many children?

\[
\frac{11}{3} = 3 \frac{2}{3} \quad \lfloor 3.67 \rfloor = 4 \quad \text{children assigned}
\]
Pigeonhole Principle – More generally

If there are \( n \) pigeons in \( k < n \) holes, then one hole must contain at least \( \frac{n}{k} \) pigeons!

**Proof.** Assume there are \( < \frac{n}{k} \) pigeons per hole. Then, there are \( < k \frac{n}{k} = n \) pigeons overall. Contradiction!
Pigeonhole Principle – Better version

If there are $n$ pigeons in $k < n$ holes, then one hole must contain at least $\left\lfloor \frac{n}{k} \right\rfloor$ pigeons!
Pigeonhole Principle – Better version

If there are $n$ pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

Reason. Can’t have fractional number of pigeons

Syntax reminder:
- Ceiling: $[x]$ is $x$ rounded up to the nearest integer (e.g., $[2.731] = 3$)
- Floor: $[x]$ is $x$ rounded down to the nearest integer (e.g., $[2.731] = 2$)
Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

Solution:
1. **367 pigeons** = people
2. **365 holes** = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday
Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps:

1. Identify pigeons
2. Identify pigeonholes
3. Specify a rule for assigning pigeons to pigeonholes
4. Apply PHP
Pigeonhole Principle – Example (Surprising?)

In every set \( S \) of 100 integers, there are at least two elements whose difference is a multiple of 37.

When solving a PHP problem:
1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

\[
\begin{align*}
n &= 100 \\
k &= 37 \\
\left\lfloor \frac{n}{k} \right\rfloor &= 3 \\
i \mod 37 &= j \mod 37 \\
i - j &= 0 \mod 37
\end{align*}
\]
Agenda

- Pigeonhole Principle
- More practice with counting
Quick Review of Cards

- 52 total cards
- 13 different ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

How many possible 5 card hands?

\[ \binom{52}{5} \]
A **straight** is five consecutive rank cards of any suit. How many possible straights?

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- #choices for rank of least card in straight: $10 \div 4 = 5$
A **flush** is a five card hand all of the same suit. How many possible flushes?

- 52 total cards
- 13 different ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
- 4 different suits: Hearts, Diamonds, Clubs, Spades
Counting Cards III

A **flush** is five card hand all of the same suit. How many possible flushes?

\[ 4 \cdot \binom{13}{5} = 5148 \]

How many flushes are **NOT** straights?

\[ \# \text{ flushes} - \# \text{ straight flushes} \]

\[ 4 \cdot 10 \]

- suit choices
- for rank of lowest card
Counting Cards III

- A flush is five card hand all of the same suit. How many possible flushes?

\[ 4 \cdot \binom{13}{5} = 5148 \]

- How many flushes are NOT straights?

\[ = \#\text{flush} - \#\text{flush and straight} \]

\[ \left( 4 \cdot \binom{13}{5} = 5148 \right) - 10 \cdot 4 \]
Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

No sequence ➔ under counting  Many sequences ➔ over counting
Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

No sequence $\rightarrow$ under counting  Many sequences $\rightarrow$ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards. $\binom{4}{3} \cdot \binom{49}{2}$
For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

\[
\binom{4}{3} \cdot \binom{49}{2}
\]

Poll:
A. Correct
B. Overcount
C. Undercount

https://pollev.com/annakarlin185
Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the \textit{unique} sequence of choices that led to it.

No sequence $\rightarrow$ under counting \hspace{1cm} Many sequences $\rightarrow$ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

When in doubt, break up into disjoint sets you know how to count, and then use the sum rule.
For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence $\Rightarrow$ under counting  Many sequences $\Rightarrow$ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

$= \# 5$ card hand containing exactly 3 Aces

$= \# 5$ card hand containing exactly 4 Aces
8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column?

Poll:
A. \( \binom{64}{3} \)
B. \( \binom{8}{3} \cdot \binom{8}{3} \)
C. \( 8^2 \cdot 7^2 \cdot 6^2 \)
D. I don’t know.

https://pollev.com/ annakarlin185
8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column?

Sequential process:
1. Column for pawn
2. Row for pawn
3. Column for bishop
4. Row for bishop
5. Column for knight
6. Row for knight

\[(8 \cdot 7 \cdot 6)^2\]
Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don’t share a row or a column

Poll:
A. $8^2 \cdot 7^2$
B. $\binom{8}{2} \cdot \binom{8}{2}$
C. $\frac{8^2 \cdot 7^2}{2}$
D. I don’t know.

https://pollev.com/annakarlin185
Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don’t share a row or a column

Pretend Rooks are different

1. Column for rook1
2. Row for rook1
3. Column for rook2
4. Row for rook2

Remove the order between two rooks

\[
\frac{(8 \cdot 7)^2}{2}
\]
Random Picture
You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, Plain.

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

**Product Rule:** In a sequential process, there are
- $n_1$ choices for the first step,
- $n_2$ choices for the second step (given the first choice), …, and
- $n_m$ choices for the $m^{th}$ step (given the previous choices),
then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, Plain

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

Try 1: First choose # of Chocolate, then # Lemon, then # Maple, then # Glazed, then # Plain

Product Rule: In a sequential process, there are
• $n_1$ choices for the first step,
• $n_2$ choices for the second step (given the first choice), …, and
• $n_m$ choices for the $m^{th}$ step (given the previous choices),

then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, Plain.

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

Try 2: First choose type of donut 1, then type of donut 2, ..., then type of donut 12.

Product Rule: In a sequential process, there are
• $n_1$ choices for the first step,
• $n_2$ choices for the second step (given the first choice), ..., and
• $n_m$ choices for the $m$th step (given the previous choices),
then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$
Bijection Rule

If there is a bijection (one-to-one and onto mapping) between set A and set B, then \(|A| = |B|\).

There is a bijection between sequences of length 16 with 4 bars and 12 stars and the doughnut choices.
doughnuts

You go to Top Pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, plain.

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

There is a bijection between sequences of length 16 with 4 bars and 12 stars and the doughnut choices.
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, plain

How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, plain

How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?

Mental process:
1. Place one donut in each flavor bin
2. Choose the remaining 7 donuts without restriction

$$\binom{7 + 5 - 1}{5 - 1}$$
Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Stars and bars
Counting is NOT for kindergarteners