Lecture 28: Clustering (mixture models) + glimpse of auction theory

Slide Credit: Based on Stefano Tessaro’s slides for 312 19au incorporating ideas from myself 😊
Agenda

• Mixture models and clustering

• A glimpse of auction theory
Motivating application: Clustering images

Discover groups of similar images
- Ocean
- Pink flower
- Dog
- Sunset
- Clouds
- ...
Motivates probabilistic model: Mixture model

- Take uncertainty in assignment into account
e.g., when clustering documents, might want to say
54% chance document is world news, 45% science,
1% sports, and 0% entertainment

- Allow for cluster shapes not just centers

- Enables learning different weightings of dimensions
  – e.g., how much to weight each word in the vocabulary when computing
  cluster assignment
Combination of weighted Gaussians

Associate a weight $\pi_k$ with each Gaussian component

$\pi = [0.47, 0.26, 0.27]$  

Fraction of each class in world from which we get data

$0 \leq \pi_k \leq 1$  

$\sum_{k=1}^{K} \pi_k = 1$
Mixture of Gaussians (1D)

Each mixture component represents a unique cluster specified by: $\{\pi_k, \mu_k, \sigma_k\}$.
Mixture of Gaussians (general)

Each mixture component represents a unique cluster specified by:

\[ \{ \pi_k, \mu_k, \Sigma_k \} \]
Mixture model

- K clusters, defined by the following unknown parameters

\[ \Theta = \{ \pi_j, \mu_j, \Sigma_j \}_{j=1}^{k} \]

\[ \sum_{j=1}^{k} \pi_j = 1. \]

- Problem: Assume that the data comes from such a distribution, and recover the parameters of the distribution (e.g. MLE)
- Determine, for each point, the likelihood of it belonging to cluster j, for each j.
Two 1-D Gaussians, with unknown mean and variance

- Easy if know the source of each data point.
Two 1-D Gaussians, with unknown mean and variance

- Easy if know the source of each data point.
- What if we don’t know the source?

\[
\text{prob}(x_i \text{ is yellow}) = \frac{f_y(x_i; \mu, \sigma^2_y)}{f_y(x_i; \mu, \sigma^2_y) + f_B(x_i; \mu_0, \sigma^2_B)}
\]

\( \mu_0 \) and \( \sigma^2_B \)
Mixture model

• K clusters, defined by the following parameters

\[ \Theta = \{ \pi_j, \mu_j, \Sigma_j \}_{j=1}^k \]

\[ \sum_{j=1}^k \pi_j = 1. \]

• Problem: Assume that the data comes from such a distribution, and recover the parameters of the distribution.

• Determine, for each point, the likelihood of it belonging to cluster j, for each j.

• **PROBLEM: no closed form solution**
Expectation Maximization Algorithm
Two step approach based on following observation

• If we knew which cluster each sample was from, we could estimate all the parameters.

• If we knew all the parameters we could estimate the chance each point came from each cluster.

• EM is an iterative algorithm that alternates between these two steps.
Agenda

• Mixture models and clustering

• A glimpse of auction theory
Auctions

- Some goods on eBay and Amazon are sold via auction.
- Companies like Google and Facebook make most of their money by selling ads.
- The ads are sold via auction.
  - Advertisers submit bids for certain “keywords”

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**Facebook Ads bidding… 😊 Is this an auction?**

Yes! That’s the first thing you need to understand to master bidding management of Facebook Ads. When you’re creating a new campaign, you’re joining a huge, worldwide auction.

You’ll be competing with hundreds of thousands of advertisers to buy what Facebook is selling: Real estate on the News Feed, Messenger, Audience Network, and mobile apps to display your ads to the users.
hawaii vacation

About 2,670,000,000 results (0.79 seconds)

Ad · https://www.expedia.com/

Hawaii Island Packages - Book with Expedia and Save

Package Deals
Today's Best Flight + Hotel Deals.
Only with Your #1 Leader in Travel.

Last Minute Deals
Expedia Last Minute Travel Deals.
Book Today, Travel Tomorrow.

Ad · https://www.airbnb.com/

Hawaii Vacation - Book & Save on Airbnb - airbnb.com
An auction is a ...

- Game
  - Players: advertisers
  - Strategy choices for each player: possible bids
  - Rules of the game – made up by Google/Facebook/whoever is running the auction

- What do we expect to happen? How do we analyze mathematically?
Special case: Sealed bid single item auction

- Say I decide to run an auction to sell my laptop and I let you be the bidders.
- If I want to make as much money as possible – what should I choose as the rules of the auction?
Special case: Sealed bid single item auction

- Say I decide to run an auction to sell my laptop and I let you be the bidders.
- If I want to make as much money as possible – what should the rules of the auction be?

Some possibilities:
- **First price auction:** highest bidder wins; pays what they bid.
- **Second price auction:** highest bidder wins; pays second highest bid.
- **All pay auction:** highest bidder wins; all bidders pay what they bid.

Which of these will make me the most money?
Special case: Sealed Bid single item auction

Some possibilities:
- **First price auction**: highest bidder wins; pays what they bid.
- **Second price auction**: highest bidder wins; pays second highest bid.
- **All pay auction**: highest bidder wins: all bidders pay what they bid.

Bidder 1

<table>
<thead>
<tr>
<th>Bidder</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bids</td>
<td>100</td>
<td>81</td>
<td>35</td>
<td>24</td>
</tr>
</tbody>
</table>

Payments

- **1st price**: 100, 0, 0, 0, 0
- **2nd price**: 81, 0, 0, 0, 0
- **All pay**: 100, 81, 35, 24
Bidder model

Each bidder has a value, say $v_i$ for bidder $i$.

Bidder is trying to maximize their “utility” – the value of the item they get – price they pay.
Theorem

A second price auction is **truthful**. In other words, it is always in each bidder’s best interest to bid their true value.
Bayes-Nash equilibrium

Suppose that $V_1 \sim F_1, V_2 \sim F_2, \ldots, V_n \sim F_n$. A bidding strategy $\beta_i(\cdot)$ is a Bayes-Nash equilibrium if $\beta_i(v_i)$ is a best response in expectation to $\beta_j(V_j) \forall j \neq i$. 
I first price. 

\[ F(x) = \frac{x}{100} \]

\[ v_a \sim U[0, 100] \]

\[ p(v_a) = \frac{v_a}{100} \]

\[ \frac{v}{3} \]

\[ E(\text{utility}) = (v - b) \Pr(\text{I win}) \]

\[ = (v - b) \Pr(b > \frac{v}{3}) \]

\[ = (v - b) \left( 1 - \frac{v}{100} \right) \]

\[ = (v - b) \frac{b}{50} \]

Choose \( b \) to max. my exp. utility.
\[ b = \frac{v}{2} \]

<table>
<thead>
<tr>
<th></th>
<th>2nd price</th>
<th>1st price</th>
<th>All pay</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V )</td>
<td>( v )</td>
<td>( \frac{v}{2} )</td>
<td>( \frac{v^2}{200} )</td>
</tr>
</tbody>
</table>

Exp anchors' revenue:

\[
E[\min(V_1, V_2)] \quad E[\max(V_1, V_2)] \quad E[\frac{V_1^2}{200} + \frac{V_2^2}{200}]
\]
Revenue Equivalence Theorem

In equilibrium, no matter what distribution the bids are drawn from, the expected auctioneer revenue is the same in all three auctions!