CSE 312

Foundations of Computing II

Lecture 17: CLT and Polling

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Anna R. Karlin

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

quiz out Monday 6pm on canvas

The Normal Distribution

Definition. A **Gaussian (or normal) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

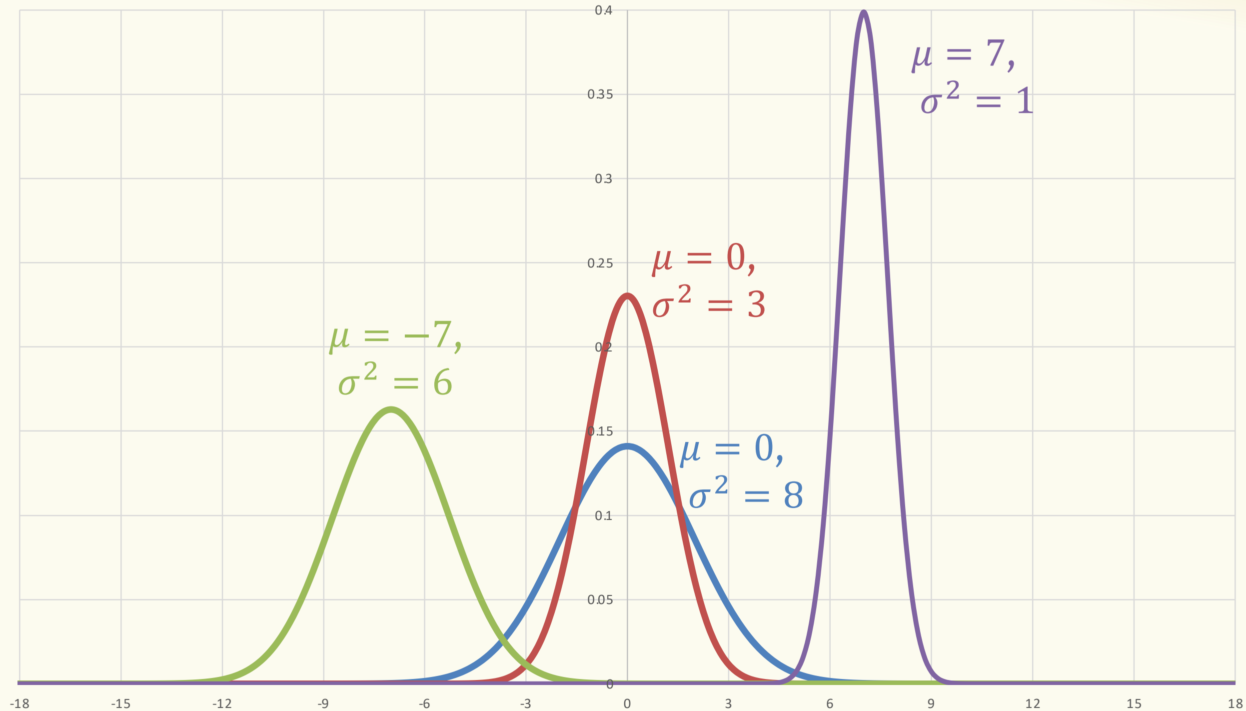
(We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Proof is easy because density curve is symmetric around μ, $f_X(\mu - x) = f_X(\mu + x)$

The Normal Distribution

Aka a "Bell Curve" (imprecise name)



CDF of normal distribution

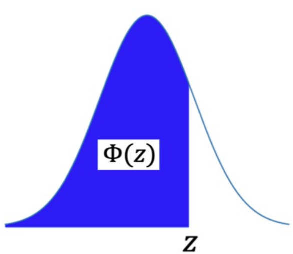
Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Standard (unit) normal $\boldsymbol{Z}$ $\sim \mathcal{N}(0, 1)$

CDF. $\Phi(z) = \mathbb{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0, 1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

Table of $\Phi(z)$ CDF of Standard Normal Distn



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

What about Non-standard normal?

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Therefore,

$$F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$$

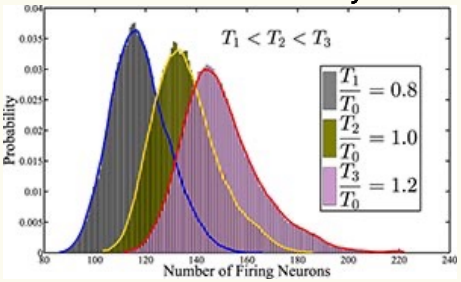
$Z \sim N(0,1)$

Agenda

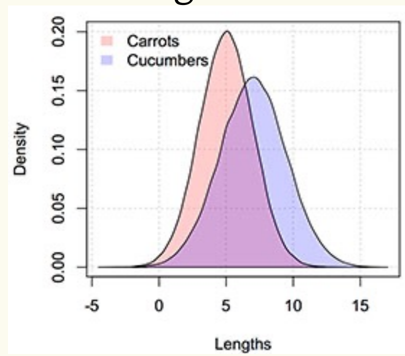
- Central Limit Theorem (CLT)
- Polling

CLT → empirical distribution of data often Gaussian

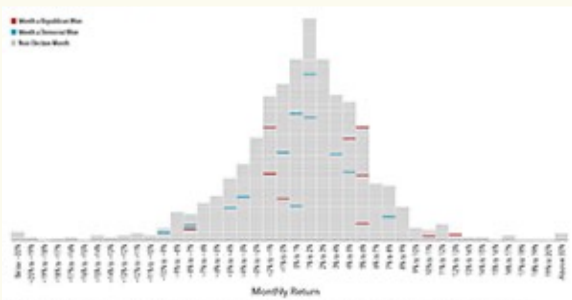
Neuron Activity



Vegetables



S&P 500 Returns after Elections



Examples from:
https://galtonboard.com/probabilityexamplesinlife

Sum of Independent RVs

i.i.d. = independent and identically distributed

$X_1, \dots, X_n$ i.i.d. with expectation μ and variance σ^2

Define

$$S_n = X_1 + \cdots + X_n$$

$$\mathbb{E}(S_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = n\mu$$

$$\mathrm{Var}(S_n) \underset{\text{indep.}}{=} \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$$

Empirical observation: S_n looks like a normal RV as n grows.

Setup for Central Limit Theorem

$X_1, \ldots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}(Y_n) = \frac{1}{\sigma\sqrt{n}}(\mathbb{E}(S_n) - n\mu) = \frac{1}{\sigma\sqrt{n}}(n\mu - n\mu) = 0$$

$$\mathrm{Var}(Y_n) = \frac{1}{\sigma^2 n}\big(\mathrm{Var}(S_n - n\mu)\big) = \frac{\mathrm{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

$\underline{X}$

$$\frac{X - \mu}{\sigma}$$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}(Y_n \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x = \Phi(y)$$

$$\boxed{X_1 + X_2 + \dots + X_n \approx \mathcal{N}(n\mu, n\sigma^2)}$$

Also stated as:

- $\lim_{n\to\infty} Y_n \to \mathcal{N}(0,1)$
- $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$ where $\mu = E[X_i]$ and $\sigma^2 = Var(X_i)$

$$E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \mu \qquad Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \underbrace{Var\left(\sum X_i\right)}_{n\sigma^2}$$

$= \frac{\sigma^2}{n}$

Agenda

- Central Limit Theorem (CLT)
- Polling

Magic Mushrooms

Not that long ago, Oregonians voted on whether to legalize the therapeutic use of “magic mushrooms”.

Poll to determine the fraction of the population that will vote in favor of legalization.

- Call up a random sample of n people to ask their opinion
- Report the empirical fraction

Questions

- Is this a good estimate?
- How to choose n?

$X_1 \quad X_2 \quad \ldots \quad X_n$

$1 \quad 0 \quad 1 \quad 1 \quad \ldots \quad 0$

$n = 500$

$\#1\text{'s} = \sum X_i = 251$

$X = \frac{1}{n}\sum_{i=1} X_i = \frac{\quad}{500}$

Polling Accuracy

Often see claims that say

"Our poll found 80% support. This poll is accurate to within 5% with 98% probability"

Will unpack what this means and how they sample enough people to know this is true.

pN

N

Formalizing Polls

Population size N, true fraction of voting in favor p, sample size n.

Problem: We don't know p

Polling Procedure

for i = 1 … n:

1. Pick uniformly random person to call (prob: $1/N$)
2. Ask them how they will vote

$$X_i = \begin{cases} 1, & \text{voting in favor} \\ 0, & \text{otherwise} \end{cases}$$

Report our estimate of p: $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$E(X^2) - E(X)]^2$

Formalizing Polls

Population size N, true fraction of voting in favor p, sample size n.

Problem: We don't know p

Poll what type of r.v. is X_i?

	Type	$E[X_i]$	$Var(X_i)$
a.	Bernoulli	p	$p(1-p)$
b.	Bernoulli	p	p^2
c.	Geometric	p	$\frac{1-p}{p^2}$
d.	Binomial	np	$np(1-p)$

https://pollev.com/ annakarlin185

Polling Procedure

for i = 1 … n:

1. Pick uniformly random person to call (prob: $1/N$)
2. Ask them how they will vote

$$X_i = \begin{cases} 1, & \text{voting in favor} \\ 0, & \text{otherwise} \end{cases}$$

$\sum_{i=1}^{n} X_i \sim Bin(n, p)$

Report our estimate of p: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Random Variables

	E	Var
X_i	p	$p(1-p)$

What type of r.v. is X_i?

What can you say about

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$?

$$E(\bar{X}) = \frac{1}{n} \underbrace{\sum_{i=1}^{n} E(X_i)}_{np} = p$$

$$Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \underbrace{Var\left(\sum X_i\right)}_{np(1-p)}$$

https://pollev.com/ annakarlin185

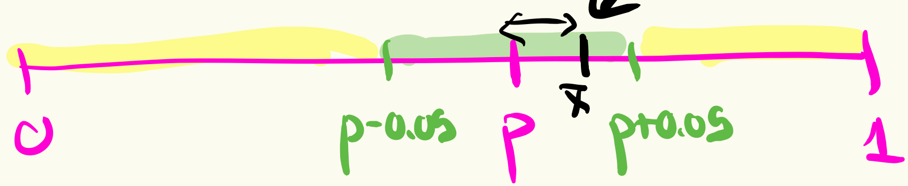
Poll:

	$E[\bar{X}]$	$Var(\bar{X})$
a.	np	$np(1-p)$
b.	p	$p(1-p)$
c.	p	$p(1-p)/n$
d.	p/n	$p(1-p)/n$

Roadmap: Bounding Error

Goal: Find the value of n such that 98% of the time, the estimate $\bar{X}$ is within 5% of the true p.

$\bar{X}$ in green good event



bad event $\bar{X}$ lands outside.

Choose n s.t.

$$\boxed{\Pr\left(|\bar{X}-p| > 0.05\right)} \le 0.02$$

in yellow

$p - 0.05 \le \bar{X} \le p + 0.05$

Central Limit Theorem

With i.i.d random variables $X_1, X_2, \ldots, X_n$ where $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$

As $n \to \infty$,

$$\frac{X_1 + X_2 + \cdots X_n \ - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1)$$

Restated: As $n \to \infty$,

$$\boxed{\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)}$$

Central Limit Theorem

Poll: In the limit $\bar{X}$ is ?

a. $\mathcal{N}(0,1)$

b. $\mathcal{N}(p,\ p(1-p)\ )$

c. $\mathcal{N}(p,\ p(1-p)/n\ )$

d. I don't know.

With i.i.d random variables $X_1, X_2, \ldots, X_n$ where $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$

As $n \to \infty$,

$$\frac{X_1 + X_2 + \cdots X_n \ - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1)$$

Restated: As $n \to \infty$,

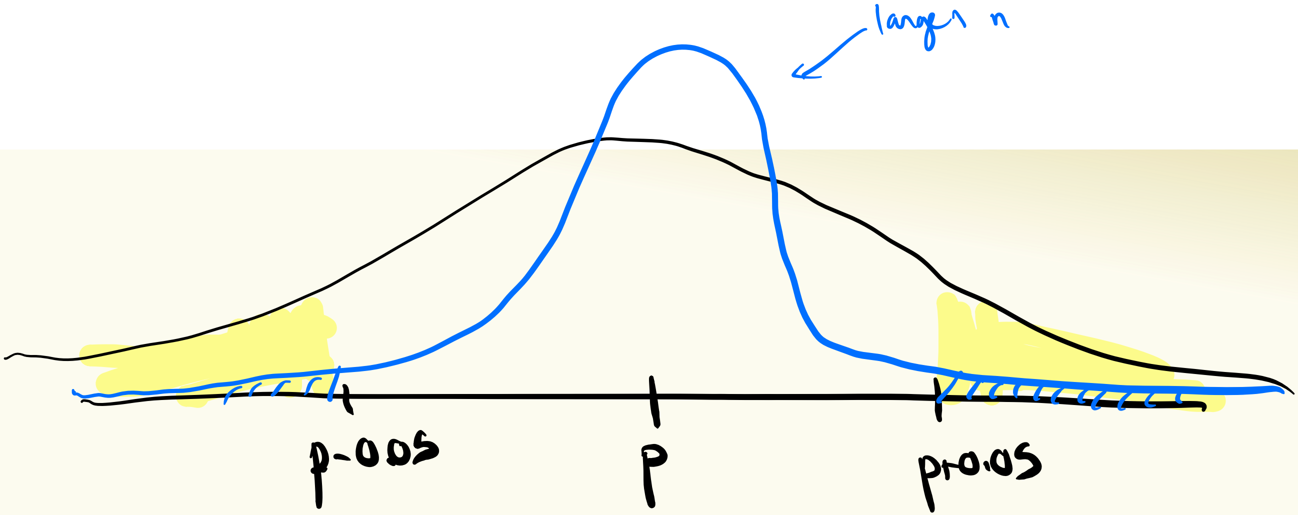
$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

$\frac{\sigma}{\sqrt{n}}$

$\forall$ normal dist

99% of prob mass

is w/in 3σ of mean.



larger n
p-0.05
p
p+0.05

Roadmap: Bounding Error

Goal: Find the value of n such that 98% of the time, the estimate $\bar{X}$ is within 5% of the true p

1. Define probability of a “bad event”
2. Apply CLT
3. Convert to a standard normal
4. Solve for n

$$\bar{X} \sim N\left(p, \frac{p(1-p)}{n}\right)$$

Define probability of a "bad event"

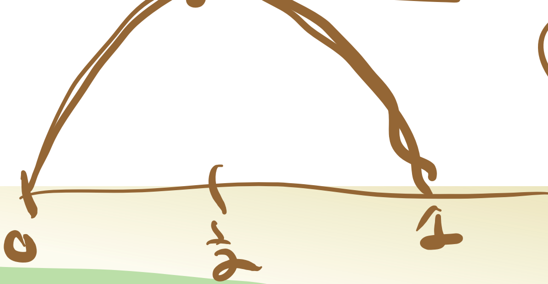
$$\Pr\left(|\bar{X}-p| > 0.05 \right)$$

$$= \Pr\left(\frac{|\bar{X}-p|}{\sqrt{\frac{p(1-p)}{n}}} > \frac{0.05}{\sqrt{\frac{p(1-p)}{n}}} \right)$$

$$= \Pr\left(|Z| > \frac{0.05\sqrt{n}}{\sqrt{p(1-p)}} \right) \le \text{brm.}$$

a

$$p(1-p) \le \frac{1}{4}$$



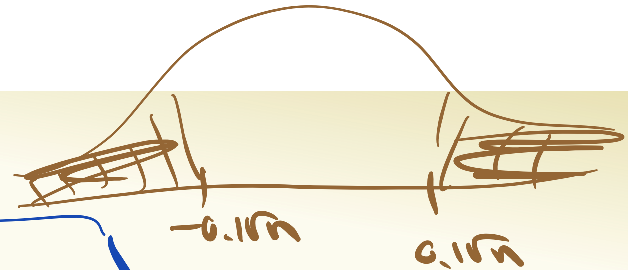
$p(1-p)$

~~Apply CLT~~

$$\frac{0.05\sqrt{n}}{\sqrt{p(1-p)}} \ge \frac{0.05\sqrt{n}}{\sqrt{\frac{1}{4}}} = 0.1\sqrt{n}$$

$$-\Pr\left(|Z| > \frac{0.05\sqrt{n}}{\sqrt{p(1-p)}}\right) \le \Pr\left(|Z| > 0.1\sqrt{n}\right)$$

[redacted]



$$\boxed{\Pr(|Z| > 0.1\sqrt{n}) < 0.02}$$

$$\Pr(Z > 0.1\sqrt{n}) + \Pr(Z < -0.1\sqrt{n})$$

$$= 2\Pr(Z > 0.1\sqrt{n})$$

$$= 2(1 - \underbrace{P(Z \le 0.1\sqrt{n})}_{\Phi(0.1\sqrt{n})}) \le 0.02$$

$$2(1 - \Phi(0.1\sqrt{n})) \le 0.02$$

$$1 - \Phi(0.1\sqrt{n}) \le 0.01$$

$$\boxed{0.99 \le \Phi(0.1\sqrt{n})}$$

Solve for n

≥ 2.33 $\quad Pr(Z \leq 2.33) = 0.99$

$0.1\sqrt{n} \geq 2.33$

$\sqrt{n} \geq \frac{2.33}{0.1}$

$$\boxed{n \geq \left(\frac{2.33}{0.1}\right)^2 = 543}$$

$$\bar{X} \sim N\left(p, \frac{p(1-p)}{n}\right)$$

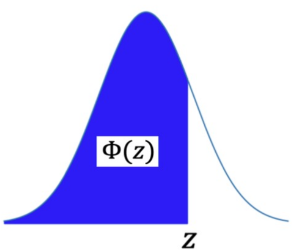
for what n is

$$Pr(|\bar{X} - p| > 0.05) < 0.02$$

$\Downarrow N(0,1)$

$p(1-p) \leq \frac{1}{4}$

Table of $\Phi(z)$ CDF of Standard Normal Distn



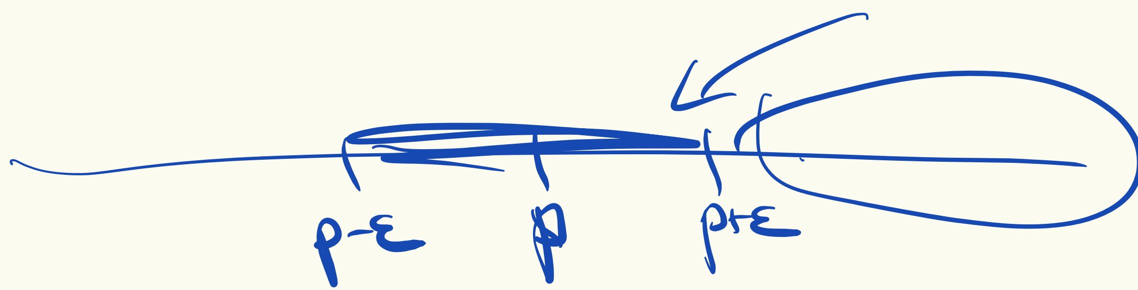
Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

$\Pr(Z \leq 2.33) = 0.99$

[redacted]

$$\Pr(|\bar{X}-p| > \underset{0.05}{\varepsilon}) \le \underset{0.02}{\delta}$$



Idealized Polling

So far, we have been discussing "idealized polling". Real life is normally not so nice ☹

Assumed we can sample people uniformly at random, not really possible in practice

- Not everyone responds
- Response rates might differ in different groups
- Will people respond truthfully?

Makes polling in real life much more complex than this idealized model!