CSE 312

Foundations of Computing II

Lecture 16: The Normal Distribution; CLT

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Anna R. Karlin

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Quiz out Monday 6pm
on canvas

Review – Continuous RVs

Probability Density Function (PDF).

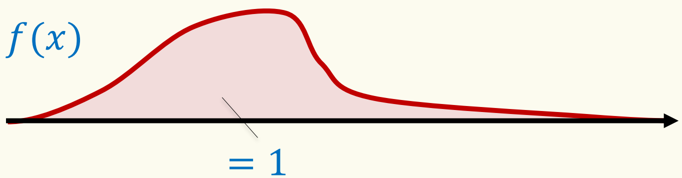
$f : \mathbb{R} \to \mathbb{R}$ s.t.

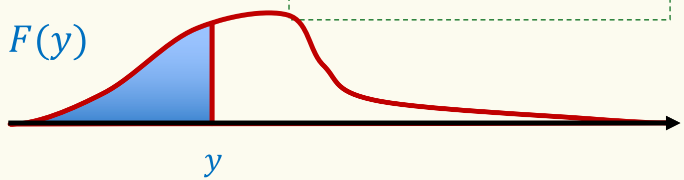
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$

Cumulative Density Function (CDF).

$$F(y) = \int_{-\infty}^{y} f(x)\,\mathrm{d}x$$

Theorem. $f(x) = \frac{dF(x)}{dx}$

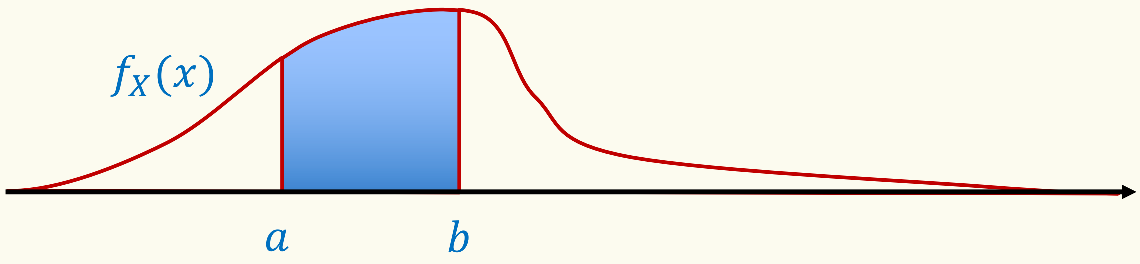




Density ≠ Probability !

$F(y) = \mathbb{P}(X \leq y)$

Review – Continuous RVs



$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(x)\mathrm{d}x = F_X(b) - F_X(a)$$

$X \sim \text{Exp}(\lambda)$

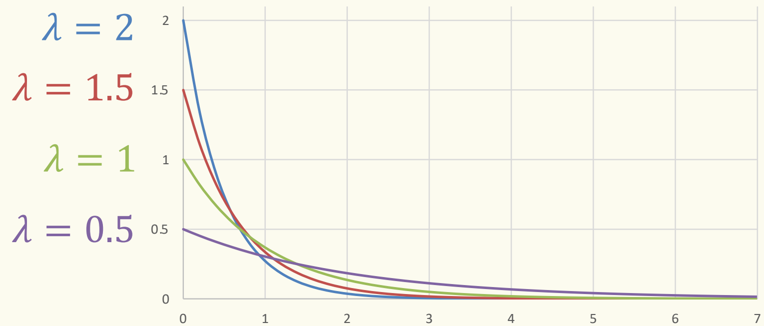
Exponential Distribution

Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \text{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$



$\Pr(X < 2) = 1 - e^{-2}$

Agenda

- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT)

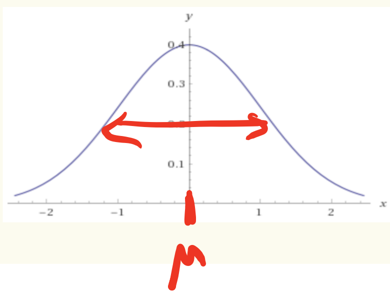
The Normal Distribution

Carl Friedrich Gauss

Definition. A **Gaussian (or normal) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

(We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)



$$\begin{cases} \mu & \text{mean} \\ \sigma^2 & \text{variance.} \end{cases}$$

The Normal Distribution

Carl Friedrich Gauss

Definition. A **Gaussian (or <u>normal</u>) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)

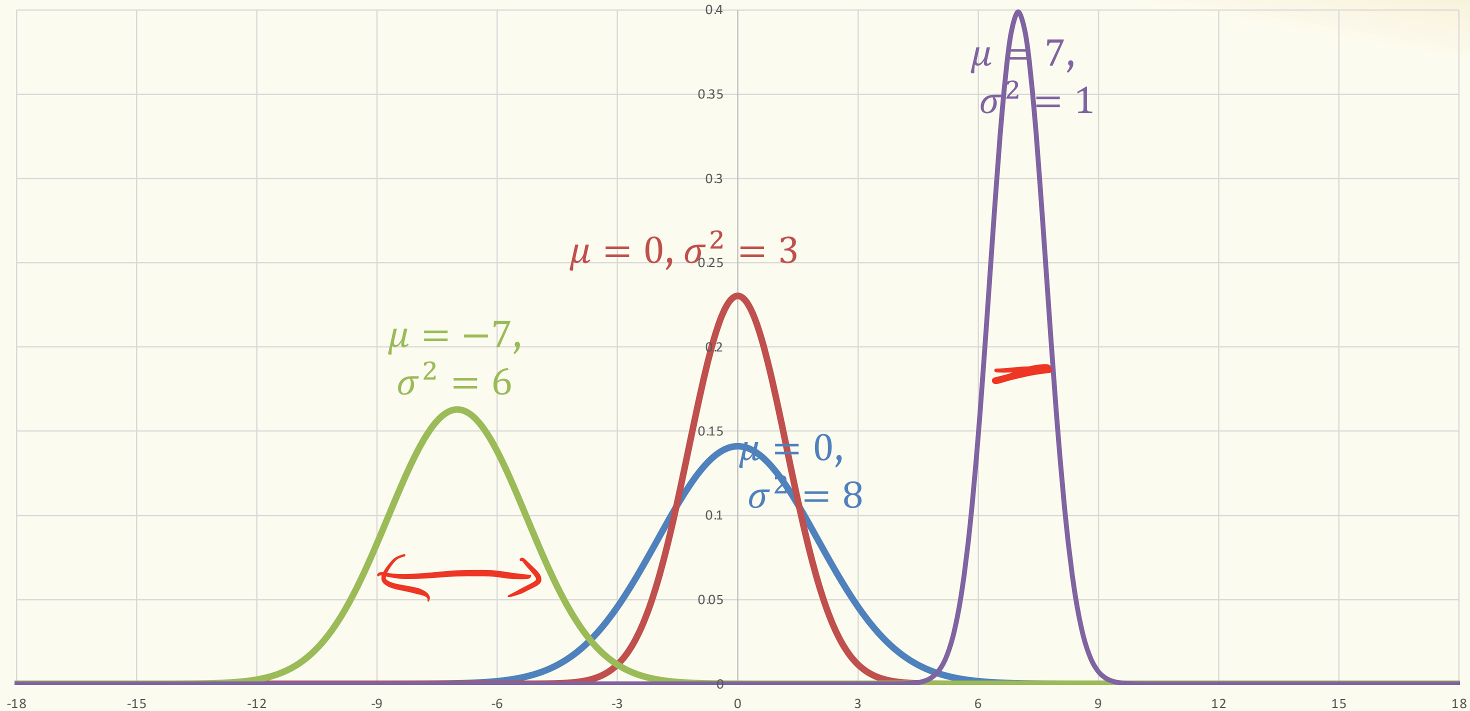
Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Expectation follows from density being symmetric around μ, $f_X(\mu - x) = f_X(\mu + x)$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

The Normal Distribution

Aka a "Bell Curve" (imprecise name)



X $\qquad$ $aX+b$

Shifting and Scaling – turning one normal dist into another

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Proof.
$$\mathbb{E}(Y) = a\,\mathbb{E}(X) + b = a\mu + b$$
$$\mathrm{Var}(Y) = a^2\,\mathrm{Var}(X) = a^2\sigma^2$$

Can show with algebra that the PDF of $Y = aX + b$ is still normal.

Note: $\boxed{\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)}$

standard normal

CDF of normal distribution

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Standard (unit) normal $\boxed{Z} \sim \mathcal{N}(0, 1)$

CDF. $\Phi(z) = \mathbb{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0, 1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

$Pr(Z \le -1.35)$

$z \ge 0$

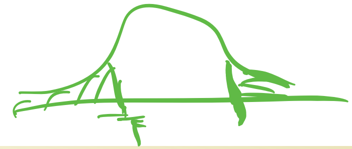
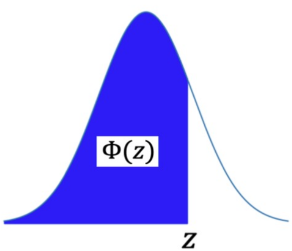


Table of $\Phi(z)$ CDF of Standard Normal Distn

$Pr(Z \le 1.35) = 0.91149$

$Pr(Z \le 2.31) = 0.98956$



Φ Table: $\mathbb{P}(Z \le z)$ when $Z \sim \mathcal{N}(0,1)$

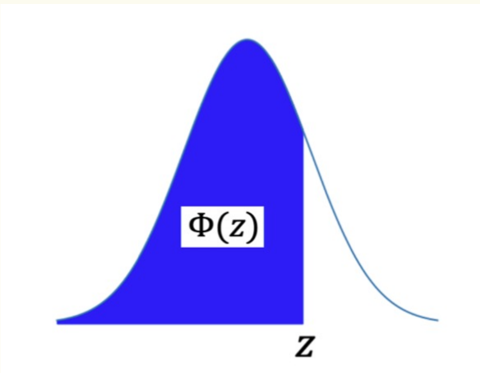
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

$Pr(Z \le 3.09) = 0.999$

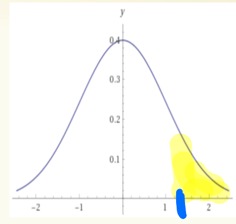
$\Pr(Z \le 1.35) = 0.91149$

$\Pr(X \le a) = \Pr(X < a)$

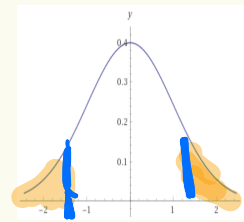
The Standard Normal CDF



$\Pr(Z \ge 1.35) = 1 - \Pr(Z < 1.35)$

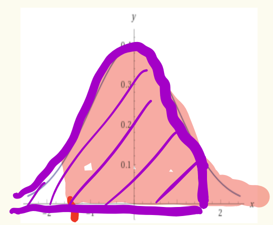


$\Pr(Z \le -1.35) = 1 - \Pr(Z < 1.35)$



−1.35

$\Pr(Z > -1.35) = \Pr(Z \le 1.35)$



−1.35

Agenda

- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT)

$$X \sim N(\mu, \sigma^2) \qquad \frac{X-\mu}{\sigma} \sim N(0,1)$$

Example

$\mu = 0.4$

$\sigma^2 = 4$

$\sigma = 2$

Let $X \sim \mathcal{N}(0.4, 4)$.

$$\mathbb{P}(X \leq 1.2) = \Pr(X - 0.4 \leq 1.2 - 0.4)$$

$$= \Pr\left(\frac{X-0.4}{2} \leq \frac{1.2-0.4}{2}\right)$$

$$Z \sim N(0,1)$$

$$= \Pr(Z \leq 0.4) = 0.65542$$

Example

Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$$\mathbb{P}(X \le 1.2) = \mathbb{P}\left(\frac{X-0.4}{2} \le \frac{1.2-0.4}{2}\right)$$

$$= \mathbb{P}\left(\frac{X-0.4}{2} \le 0.4\right) = \Phi(0.4) \approx 0.6554$$

$\sim \mathcal{N}(0,1)$

0.1	0.5398	0.5438
0.2	0.5793	0.5832
0.3	0.6179	0.6217
0.4	0.6554	0.6591
0.5	0.6915	0.6950
0.6	0.7257	0.7291
0.7	0.7580	0.7611

Example

$\mu = 3$
$\sigma^2 = 16$
$\sigma = 4$

Let $X \sim \mathcal{N}(3, 16)$.

$\mathbb{P}(2 < X < 5)$

$$= \Pr\left(\frac{2-3}{4} < \frac{X-3}{4} < \frac{5-3}{4}\right)$$

$Z \sim N(0,1)$

$$= \Pr\left(-\frac{1}{4} < Z < \frac{1}{2}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right) = \Phi\left(\frac{1}{2}\right) + \Phi\left(\frac{1}{4}\right) - 1$$

Example

Let $X \sim \mathcal{N}(3, 16)$.

$$\mathbb{P}(2 < X < 5) = \mathbb{P}\left(\frac{2-3}{4} < \frac{X-3}{4} < \frac{5-3}{4}\right)$$

$$= \mathbb{P}\left(-\frac{1}{4} < Z < \frac{1}{2}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right)$$

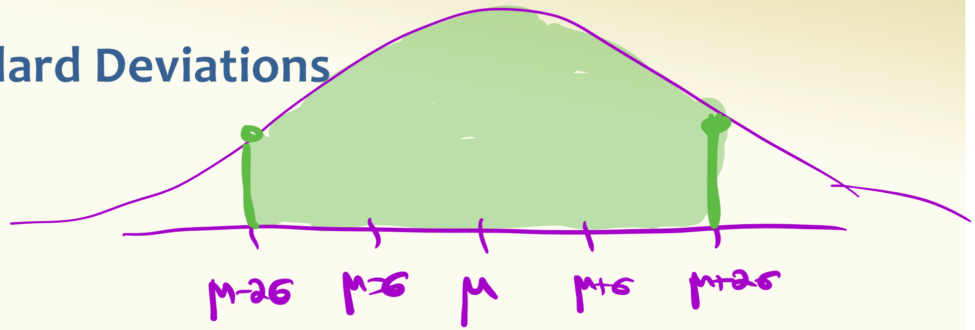
$$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right) \approx 0.29017$$

Example – Off by Standard Deviations

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$\mathbb{P}(|X - \mu| < k\sigma) =$

2



$k = 2$

$\Pr(|X-\mu| < 2\sigma)$

$$\Pr(-k\sigma < X - \mu < k\sigma)$$

$$= \Pr\left(-k < \frac{X-\mu}{\sigma} < k\right)$$

$$= \Pr(-k < Z < k)$$

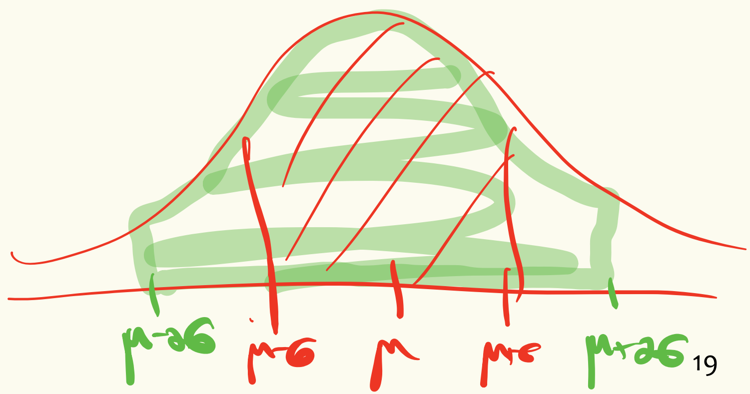
Example – Off by Standard Deviations

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$\mathbb{P}(|X-\mu| < k\sigma) = \mathbb{P}\left(\frac{|X-\mu|}{\sigma} < k\right) =$$

$$= \mathbb{P}\left(-k < \frac{X-\mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)$$

e.g. $k = 1$: 68%, $k = 2$: 95%, $k = 3$: 99%



Summary of procedure for doing calculations with normal r.v.

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

Therefore,

$$F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$$

Z

CDF of normal distribution

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Standard (unit) normal $\boldsymbol{Z} \sim \mathcal{N}(0,1)$

CDF. $\Phi(z) = \mathbb{P}(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0,1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F_X(z) = \mathbb{P}(X \le z) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \le \frac{z-\mu}{\sigma}\right) = \Phi(\frac{z-\mu}{\sigma})$

$X \sim N(\mu, \sigma^2)$ $\quad$ $Y = -X \sim N(-\mu, \sigma^2)$

Closure of the normal -- under addition

Fact. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV) then $aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that the sum of normal **RVs is still a normal RV.**

The values of the expectation and variance is not surprising.

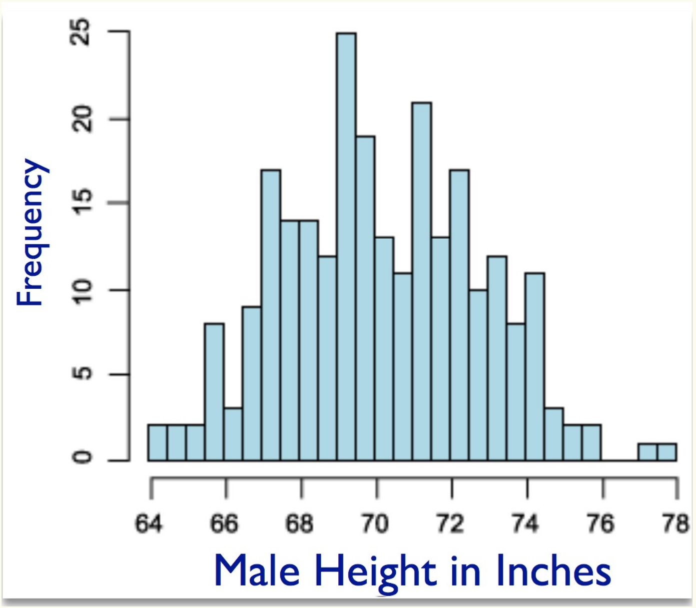
- Linearity of expectation (always true)
- When X and Y are independent, $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$

Agenda

- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT)

Gaussian in Nature

Empirical distribution of collected data often resembles a Gaussian …



e.g. Height distribution resembles Gaussian.

R.A.Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can written as

$$X = X_1 + \cdots + X_n$$

Sum of Independent RVs

i.i.d. = independent and identically distributed

$X_1, \dots, X_n$ i.i.d. with expectation μ and variance σ^2

Define

$$\boxed{S_n} = X_1 + \cdots + X_n$$

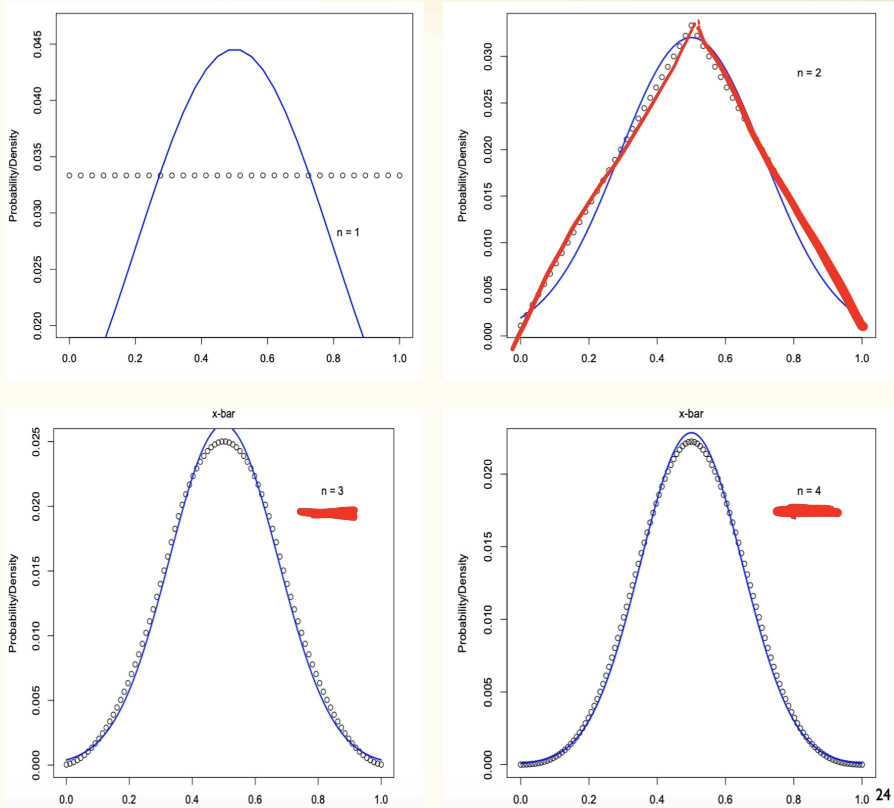
$$\mathbb{E}(S_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = n\mu$$

$$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$$

indep r.v.,

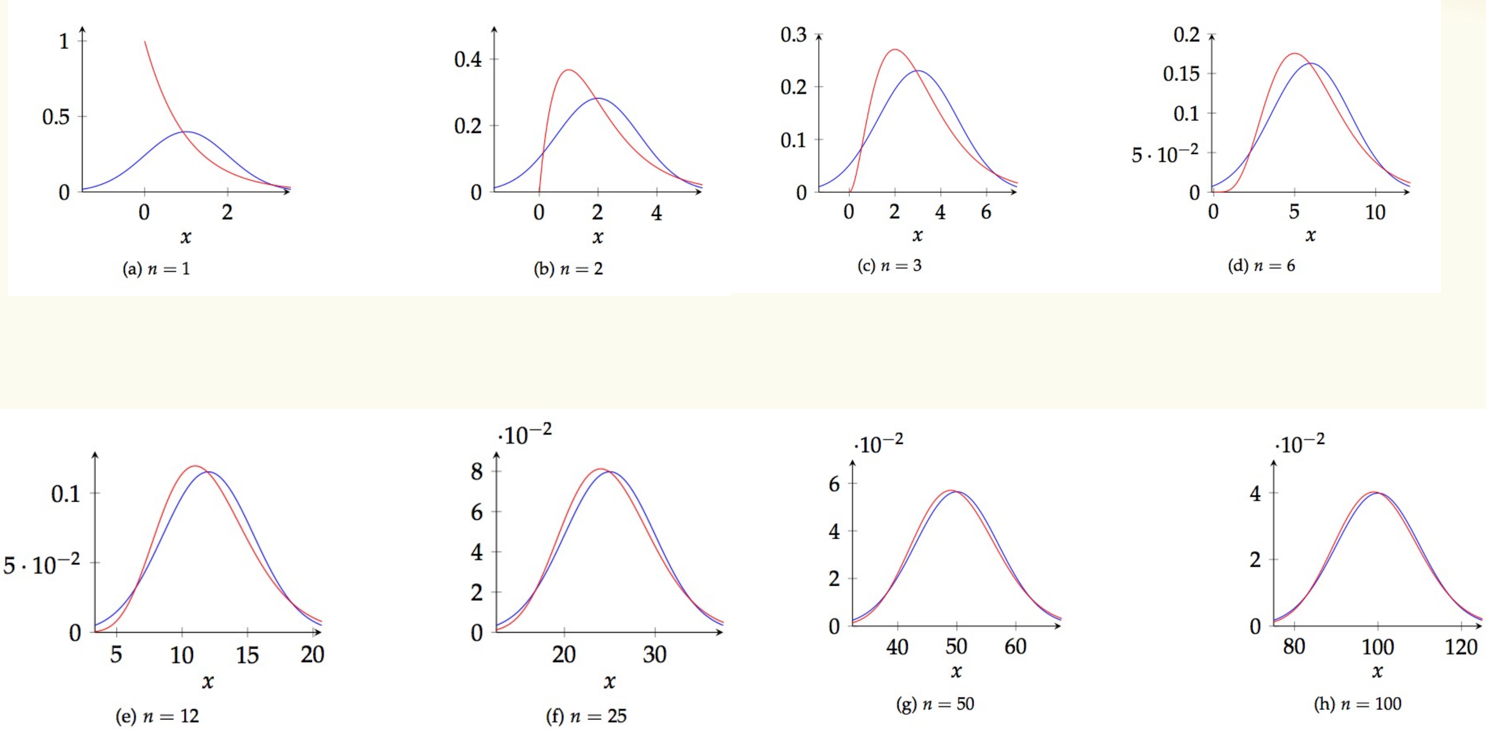
Empirical observation: S_n looks like a normal RV as n grows.

CLT (Idea)



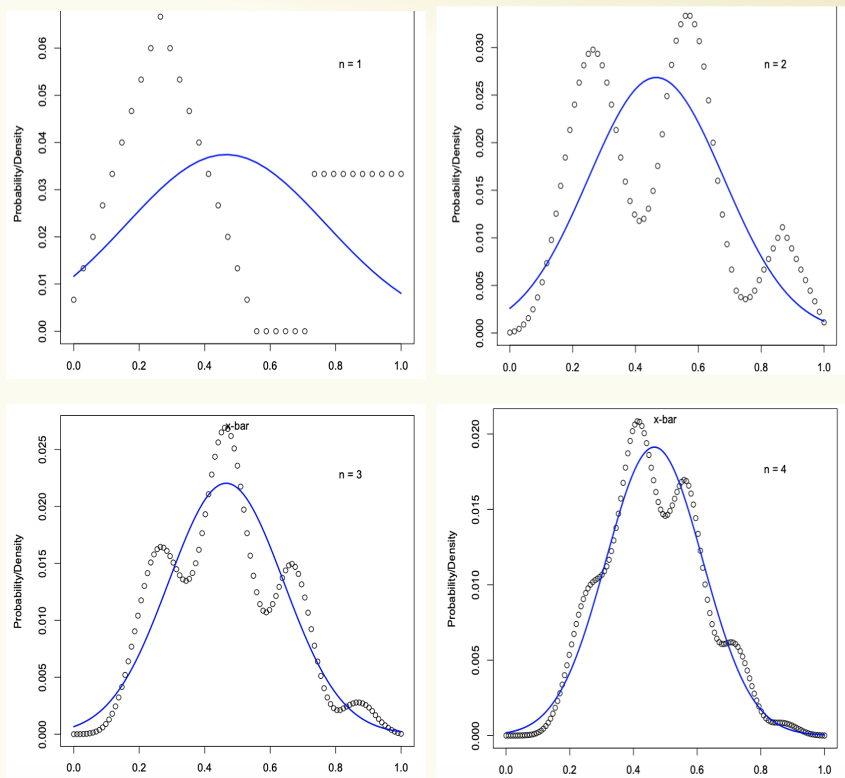
From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf

Sum of i.i.d. exponential random variables (param 1)



$$\Pr(X+Y=6)=\sum_{i=1}^{5}\Pr(X=i,\,Y=6-i)$$

CLT (Idea)



From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf

Central Limit Theorem

$X_1, \dots, X_n$ i.i.d., each with expectation μ and variance σ^2

$E(S_n) = n\mu$

$\boxed{\mathrm{Var}(S_n) = n\sigma^2}$

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$Y_n = \frac{S_n - E(S_n)}{\sigma(S_n)}$$

$\mathrm{stddev}(S_n) = \sqrt{n}\,\sigma$

$$\mathbb{E}(Y_n) = 0$$

$$\mathrm{Var}(Y_n) = 1$$

S_n r.v.

$$\frac{X-\mu}{\sigma}$$

$$E\left(\frac{X-\mu}{\sigma}\right) = 0$$

$$V\left(\frac{X-\mu}{\sigma}\right) = 1$$

Central Limit Theorem

$X_1, \ldots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}(Y_n) = \frac{1}{\sigma\sqrt{n}}\left(\mathbb{E}(S_n) - n\mu\right) = \frac{1}{\sigma\sqrt{n}}(n\mu - n\mu) = 0$$

$$\mathrm{Var}(Y_n) = \frac{1}{\sigma^2 n}\left(\mathrm{Var}(S_n - n\mu)\right) = \frac{\mathrm{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

Central Limit Theorem

finite mean
pos finite var

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}(Y_n \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x = \Phi(y)$$

$$\boxed{= \Pr(Z \le y)}$$

$\mathcal{N}(0,1)$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

Also stated as:

- $\boxed{\lim_{n\to\infty} Y_n \to \mathcal{N}(0,1)}$
- $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \to \mathcal{N}(\mu, \frac{\sigma^2}{n})$ where $\boxed{\mu = E[X_i]}$ and $\sigma^2 = Var(X_i)$

$E(\frac{1}{n}\sum_{i=1}^{n} X_i) = \mu$

$Var(\frac{1}{n}\sum_{i=1}^{n} X_i)$

$= \frac{1}{n^2} Var\left(\sum_{i=1} X_i\right) = \frac{1}{n^2} \cdot \ldots = \frac{\sigma^2}{n}$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

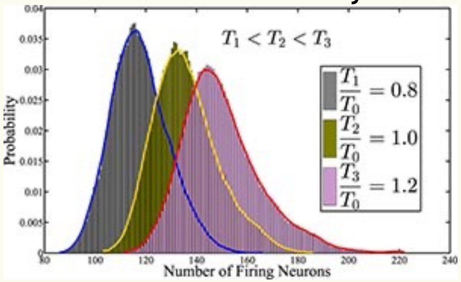
$$\lim_{n\to\infty} \mathbb{P}(Y_n \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

Also stated as:

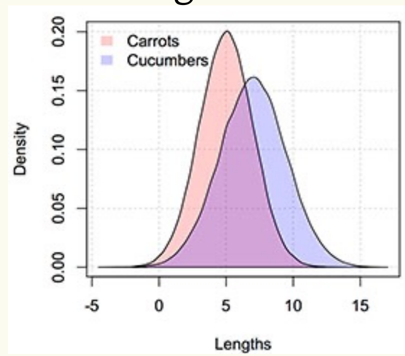
- $\lim_{n\to\infty} Y_n \to \mathcal{N}(0,1)$
- $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$ where $\mu = E[X_i]$ and $\sigma^2 = Var(X_i)$

CLT → Normal Distribution EVERYWHERE

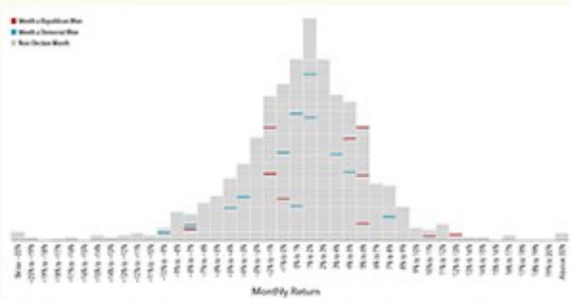
Neuron Activity



Vegetables



S&P 500 Returns after Elections



Examples from:
https://galtonboard.com/probabilityexamplesinlife