CSE 312
Foundations of Computing II

Lecture 13: The Poisson Distribution

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Slide Credit: Based on Stefano Tessaro’s slides for 312 19au incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊
### Zoo of Discrete RVs!

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Formula</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X \sim \text{Unif}(a, b) )</td>
<td>[ P(X = k) = \frac{1}{b - a + 1} ]</td>
<td>[ E[X] = \frac{a + b}{2} ]</td>
<td>[ Var(X) = \frac{(b - a)(b - a + 2)}{12} ]</td>
</tr>
<tr>
<td>( X \sim \text{Ber}(p) )</td>
<td>[ P(X = 1) = p, P(X = 0) = 1 - p ]</td>
<td>[ E[X] = p ]</td>
<td>[ Var(X) = p(1 - p) ]</td>
</tr>
<tr>
<td>( X \sim \text{Bin}(n, p) )</td>
<td>[ P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} ]</td>
<td>[ E[X] = np ]</td>
<td>[ Var(X) = np(1 - p) ]</td>
</tr>
<tr>
<td>( X \sim \text{Geo}(p) )</td>
<td>[ P(X = k) = (1 - p)^{k-1}p ]</td>
<td>[ E[X] = \frac{1}{p} ]</td>
<td>[ Var(X) = \frac{1 - p}{p^2} ]</td>
</tr>
<tr>
<td>( X \sim \text{NegBin}(r, p) )</td>
<td>[ P(X = k) = \binom{k-1}{r-1} p^r (1 - p)^{k-r} ]</td>
<td>[ E[X] = \frac{r}{p} ]</td>
<td>[ Var(X) = \frac{r(1 - p)}{p^2} ]</td>
</tr>
<tr>
<td>( X \sim \text{HypGeo}(N, K, n) )</td>
<td>[ P(X = k) = \frac{K}{N} \binom{N-K}{n-k} \binom{N}{n} ]</td>
<td>[ E[X] = \frac{K}{N} ]</td>
<td>[ Var(X) = n \frac{K(N-K)(N-n)}{N^2(N-1)} ]</td>
</tr>
</tbody>
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Agenda

• Poisson Distribution

• Approximate Binomial distribution using Poisson distribution
Preview: Poisson

Model: # events that occur in an hour
- Expect to see 3 events per hour (but will be random)
- The expected number of events in $t$ hours, is $3t$
- Occurrence of events on disjoint time intervals is independent

Example – Model cars passing through a certain town in 1 hour

$X = \# \text{ cars passing through a certain town in 1 hour}$
Divide 1 hour into $n$ intervals each of length $1/n$

What should $p$ be?

Poll:
A. $3/n$
B. $3n$
C. $3$
D. $3/60$

https://pollev.com/annakarlin185
Example – Model the process of cars passing through a light in 1 hour

\( X = \) \# cars passing through a light in 1 hour

Know: \( \mathbb{E}(X) = \lambda \) for some given \( \lambda > 0 \)

Discretize problem: \( n \) intervals, each of length \( \frac{1}{n} \).

In each interval, a car passes by with probability \( \frac{\lambda}{n} \) (assume \( \leq 1 \) car can pass by)

Bernoulli \( X_i = 1 \) if car in \( i \)-th interval (0 otherwise). \( \mathbb{P}(X_i = 1) = \frac{\lambda}{n} \)

\( X = \sum_{i=1}^{n} X_i \) \( X \sim \text{Binomial}(n,p) \) \( \mathbb{P}(X = i) = \binom{n}{i} \left( \frac{\lambda}{n} \right)^i \left( 1 - \frac{\lambda}{n} \right)^{n-i} \)

indeed! \( \mathbb{E}(X) = \lambda \)
Don’t like discretization

We want now $n \to \infty$

$\Pr(X = i) = \binom{n}{i} \left( \frac{\lambda}{n} \right)^i \left( 1 - \frac{\lambda}{n} \right)^{n-i} = \frac{n!}{(n-i)!} \frac{\lambda^i}{i!} \left( 1 - \frac{\lambda}{n} \right)^n \left( 1 - \frac{\lambda}{n} \right)^{-i}$

$\to \Pr(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \to 1 \to e^{-\lambda} \to 1$

$X$ is Binomial $\Pr(X = i) = \binom{n}{i} \left( \frac{\lambda}{n} \right)^i \left( 1 - \frac{\lambda}{n} \right)^{n-i}$
Poisson Distribution

- Suppose “events” happen, independently, at an average rate of $\lambda$ per unit time.
- Let $X$ be the actual number of events happening in a given time unit. Then $X$ is a Poisson r.v. with parameter $\lambda$ (denoted $X \sim \text{Poi}(\lambda)$) and has distribution (PMF):

  $$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Several examples of “Poisson processes”:
- # of cars passing through a certain town in 1 hour
- # of requests to web servers in a minute
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour
Probability Mass Function

\[ \mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \]

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Validity of Distribution

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Fact. $$\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$$
Validity of Distribution

We first want to verify that Poisson probabilities sum up to 1.

\[
\sum_{i=0}^{\infty} \mathbb{P}(X = i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^\lambda = 1
\]

Fact. \( \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x \)
Theorem. If $X$ is a Poisson RV with parameter $\lambda$, then

$$\mathbb{E}(X) = \lambda$$

Proof. 

$$\mathbb{E}(X) = \sum_{i=0}^{\infty} i \cdot \mathbb{P}(X = i)$$
**Theorem.** If $X$ is a Poisson RV with parameter $\lambda$, then 
\[
\mathbb{E}(X) = \lambda
\]

**Proof.** 
\[
\mathbb{E}(X) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i - 1)!} \\
= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i - 1)!} \\
= 1 \text{ (see prior slides!)}
\]

\[
= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda
\]
**Theorem.** If $X$ is a Poisson RV with parameter $\lambda$, then $\text{Var}(X) = \lambda$

**Proof.**

$$
\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda
$$
Theorem. If $X$ is a Poisson RV with parameter $\lambda$, then $\text{Var}(X) = \lambda$

Proof. $\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} \cdot i$

$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j + 1)$

$= \lambda \left[ \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j + \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \right] = \lambda^2 + \lambda$

$\mathbb{E}(X) = \lambda$  

$= 1$

Similar to the previous proof

Verify offline.

$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$
**Poisson Random Variables**

**Definition.** A Poisson random variable $X$ with parameter $\lambda \geq 0$ is such that for all $i = 0, 1, 2, 3 \ldots$,

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Poisson approximates Binomial when $n$ is very large, $p$ is very small, and $\lambda = np$ is “moderate” (e.g. $n > 20$ and $p < 0.05$, $n > 100$ and $p < 0.1$)

Formally, Binomial is Poisson in the limit as $n \to \infty$ (equivalently, $p \to 0$) while holding $np = \lambda$
From Binomial to Poisson

\[
X \sim \text{Bin}(n, p)
\]

\[
P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}
\]

\[
E[X] = np
\]

\[
Var(X) = np(1 - p)
\]

\[
X \rightarrow \infty
\]

\[
np = \lambda
\]

\[
p = \frac{\lambda}{n} \rightarrow 0
\]

\[
X \sim \text{Poisson}(\lambda)
\]

\[
P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}
\]

\[
E[X] = \lambda
\]

\[
Var(X) = \lambda
\]
Probability Mass Function – Convergence of Binomials

\( \lambda = 5 \)
\( p = \frac{5}{n} \)
\( n = 10, 15, 20 \)

\( \text{as } n \to \infty, \quad \text{Binomial}(n, \ p = \lambda/n) \to \text{pois}(\lambda) \)
Consider sending bit string over a network

- Send bit string of length \( n = 10^4 \)
- Probability of (independent) bit corruption is \( p = 10^{-6} \)
- What is probability that message arrives uncorrupted?

Using \( Y \sim \text{Bin}(10^4, 10^{-6}) \)
\[
P(Y = 0)
\]

Using \( X \sim \text{Poi}(\lambda = np = 10^4 \cdot 10^{-6} = 0.01) \)
\[
P(X = 0)
\]
Example -- Approximate Binomial Using Poisson

Consider sending bit string over a network

- Send bit string of length $n = 10^4$
- Probability of (independent) bit corruption is $p = 10^{-6}$
- What is probability that message arrives uncorrupted?

Using $Y \sim \text{Bin}(10^4, 10^{-6})$

$$\mathbb{P}(Y = 0) \approx 0.990049829$$

Using $X \sim \text{Poi}(\lambda = np = 10^4 \cdot 10^{-6} = 0.01)$

$$\mathbb{P}(X = 0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-0.01} \cdot \frac{0.01^0}{0!} = 0.990049834$$
Sum of Independent Poisson RVs

**Theorem.** Let $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = (X + Y)$. For all $k = 0,1,2,3 \ldots$,

$$
P(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}
$$

More generally, let $X_1 \sim Poi(\lambda_1), \ldots, X_n \sim Poi(\lambda_n)$ such that $\lambda = \sum_i \lambda_i$.
Let $Z = \sum_i X_i$

$$
P(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}
$$
Sum of Independent Poisson RVs

**Theorem.** Let $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = (X + Y)$. For all $k = 0,1,2,3 \ldots$, 

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$\mathbb{P}(Z = k) =$ ?

1. $\mathbb{P}(Z = k) = \sum_{j=0}^{k} \mathbb{P}(X = j, Y = k - j)$
2. $\mathbb{P}(Z = k) = \sum_{j=0}^{\infty} \mathbb{P}(X = j, Y = k - j)$
3. $\mathbb{P}(Z = k) = \sum_{j=0}^{k} \mathbb{P}(Y = k - j | X = j) \mathbb{P}(X = j)$
4. $\mathbb{P}(Z = k) = \sum_{j=0}^{k} \mathbb{P}(Y = k - j | X = j)$

**Poll:**

A. All of them are right  
B. The first 3 are right  
C. Only 1 is right  
D. Don’t know
1. $\mathbb{P}(Z = k) = \sum_{j=0}^{k} \mathbb{P}(X = j, Y = k - j)$
2. $\mathbb{P}(Z = k) = \sum_{j=0}^{\infty} \mathbb{P}(X = j, Y = k - j)$
3. $\mathbb{P}(Z = k) = \sum_{j=0}^{k} \mathbb{P}(Y = k - j | X = j) \mathbb{P}(X = j)$
4. $\mathbb{P}(Z = k) = \sum_{j=0}^{k} \mathbb{P}(Y = k - j | X = j)$
\[ \mathbb{P}(Z = k) = \sum_{j=0}^{k} \mathbb{P}(X = j, Y = k - j) \]

Law of total probability

= \sum_{j=0}^{k} \mathbb{P}(X = j) \mathbb{P}(Y = k - j)

Independence
\[ \mathbb{P}(Z = z) = \sum_{j=0}^{k} \mathbb{P}(X = j, Y = z - j) \]

Law of total probability

\[ = \sum_{j=0}^{k} \mathbb{P}(X = j) \mathbb{P}(Y = z - j) = \sum_{j=0}^{k} e^{-\lambda_1} \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \frac{\lambda_2^{z-j}}{z-j!} \]

Independence

\[ = e^{-\lambda} \left( \sum_{j=0}^{k} \frac{1}{j! z-j!} \cdot \frac{\lambda_1^j \lambda_2^{z-j}}{z!} \right) \]

\[ = e^{-\lambda} \left( \sum_{j=0}^{k} \frac{z!}{j! z-j!} \cdot \frac{\lambda_1^j \lambda_2^{z-j}}{z!} \right) \frac{1}{z!} \]

\[ = e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^z \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^z \cdot \frac{1}{z!} \]

Binomial Theorem
Poisson Random Variables

**Definition.** A Poisson random variable $X$ with parameter $\lambda \geq 0$ is such that for all $i = 0, 1, 2, 3 \ldots$,

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**General principle:**
- Events happen at an average rate of $\lambda$ per time unit
- Number of events happening at a time unit $X$ is distributed according to $\text{Poi}(\lambda)$
- Poisson approximates Binomial when $n$ is large, $p$ is small, and $np$ is moderate
- Sum of independent Poisson is still a Poisson
Next Time

- Continuous Random Variables
- Probability Density Function
- Cumulative Density Function

Often we want to model experiments where the outcome is not discrete.
Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

• $T$ = time of lightning strike

• Every time within $[0,1]$ is equally likely
  – Time measured with infinitesimal precision.

$T = 0.71237131931129576 ...$

The outcome space is not discrete
Lightning strikes a pole within a one-minute time frame

- $T =$ time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$\mathbb{P}(T \geq 0.5) =$$
Lightning strikes a pole within a one-minute time frame

- $T$ = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$\mathbb{P}(0.2 \leq T \leq 0.5) =$$
Lightning strikes a pole within a one-minute time frame

- $T$ = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$\mathbb{P}(T = 0.5) =$
Bottom line

• This gives rise to a different type of random variable
• \( \mathbb{P}(T = x) = 0 \) for all \( x \in [0,1] \)
• Yet, somehow we want
  – \( \mathbb{P}(T \in [0,1]) = 1 \)
  – \( \mathbb{P}(T \in [a, b]) = b - a \)
  – ...
• How do we model the behavior of \( T \)?