**Q1 Pigeonhole Principle (PHP)**

2 Points

What is the MINIMUM number of pigeons such that when they are placed into 7 pigeonholes, there is ALWAYS at least one pigeonhole with 3 or more pigeons in it?

*Hint: This is asking you to reverse the reasoning in the Pigeonhole Principle (PHP). As you know, the PHP says that if there are $n$ pigeons and $m$ pigeonholes ($n > m$), then there exists at least one hole with $\left\lceil \frac{n}{m} \right\rceil$ pigeons. For this question, you have to find the minimum value for $n$.***

15

**EXPLANATION**

There are two ways to think about this:

**Approach 1:** Solve for the minimum integer $n$ such that

$$\left\lceil \frac{n}{7} \right\rceil = 3.$$

This is pretty easy to eyeball. If $n = 14$, then $\frac{n}{7}$ is an integer, so it is equal to its ceiling, which is 2. If we take any integer larger than 14, the ceiling is 3. Therefore, the answer is 15.

**Approach 2:** Think about it intuitively.

First, imagine putting pigeons into holes in such a way so as to try and avoid getting at least 3 pigeons in any one hole. Wouldn't you want to spread them out as evenly as possible?

Following this process, you would eventually have no choice but to put a third pigeon in a pigeonhole. When would this happen? It would have to be when you already have 2 pigeons in every hole. Since there are 7 holes, this is when you have placed 14 pigeons. Thus, the 15th pigeon is when you have no choice but to fill a pigeonhole to three. Therefore, 15 pigeons is the minimum number of pigeons such that there will always be at least one pigeonhole with three pigeons in it.

**Q2 Are you counting correctly?**
4 Points

It is easy to make mistakes when solving counting problems. For the following scenarios, decide if the given approach is correct, or determine why it is incorrect.

Q2.1 3 card hands
1 Point

How many ways can you draw a 3-card hand from a standard 52-card deck, such that each card in the hand has a different rank (and order doesn't matter)?

Here is one approach: There are 52 ways to select the first card. Now there are 48 choices left since we can't choose any card with the same rank (number) as the first card selected. Finally, there are 44 choices left for the third card, because we can't choose a card with the same rank as either of the first two cards selected. Therefore, there are $52 \cdot 48 \cdot 44$ ways to choose the cards for a 3-card hand consisting of different numbers.

Select the correct assertion about the above approach:

- There is nothing wrong with the above approach.
- The analysis above over-counts. Because the order of the cards within a hand doesn't matter, we need to divide by the number of times each unordered set of 3 cards is selected. The correct result should be $\frac{52 \cdot 48 \cdot 44}{3!}$.
- The analysis above over-counts. Because the order of the cards within a hand doesn't matter, we need to take into account the fact that the same hand is counted multiple times (in different orders). Thus, the correct result should be $\frac{52 \cdot 48 \cdot 44}{3! \cdot 13 \cdot 4!}$.
- The analysis above under-counts. Because the order of the cards within a hand doesn't matter, we need to take into account the fact that the same hand is counted multiple times (in different orders). Thus, the correct result should be $52 \cdot 48 \cdot 44 \cdot 3!$.

EXPLANATION
Assertion 2 is correct. The analysis at the top of this part of the problem was almost correct, except it failed to take into account that the same hand is selected multiple times in different orders. To compensate for this, you have to divide by $3!$. 
Q2.2 Apples
1 Point

There are 10 indistinguishable apples. We want to distribute them all to 4 TAs: Jerome, Melissa, Anna G. and Sherry (which means the TAs are distinct). There is no requirement that this distribution be fair or that every TA will receive some apples. We want to find the total number of different ways to distribute the apples among the TAs.

One approach is to use the Stars and Bars method. Think of the 10 apples as stars and add 4 bars to represent the TAs in the order Jerome, Melissa, Anna G. and Sherry. Therefore, we simply need to choose a sequence of 14 stars and bars that has exactly 4 bars. The number of ways to do this is $\binom{10+4}{4}$.

Select the correct assertion:

- There is nothing wrong with the above analysis.
- The analysis above over-counts. Because there are 10 apples to distribute to 4 bins (the TAs), the correct answer should be $\binom{10}{4}$.
- The analysis above over-counts. There should actually only be $4 - 1$ bars to separate the 10 apples for 4 people, so the correct answer should be $\binom{10+4-1}{4-1}$.
- The analysis above under-counts. Because the order of apples and bars matters, the correct answer should be $P(10 + 4, 4)$.
EXPLANATION
Assertion 3 is the only correct answer. To divide the apples between 4 people, we only need 3 bars, because that gives 4 groups of stars (the stars to the left of the 1st bar, the stars between the 1st and 2nd, between the 2nd and 3rd, and to the right of the 3rd.) So, we need to choose a sequence of 13 stars and bars where 3 of them are bars and the correct answer is:

\[
\binom{10 + 4 - 1}{4 - 1} = \binom{13}{3}
\]

In general, the formula for distributing \( n \) indistinguishable objects across \( k \) distinguishable boxes (where some boxes can have 0 objects) is:

\[
\binom{n + k - 1}{k - 1}
\]

Try to solve the variation of this problem in which each TA must get at least one apple. You can still use the Stars and Bars method :).

Q2.3 More fruit
2 Points

There are 3 distinct boxes and 6 distinct fruits. How many different ways are there to place the fruits into these boxes such that no box is empty?

Here is one approach: There are 6 distinct fruits, and each fruit has 3 different boxes that it can be put into, so there are \( 3^6 \) ways to arrange the fruits into these boxes without any restrictions.

Next, we want to find the number of ways to arrange the fruits such that at least 1 box out of the 3 is empty. There are \( \binom{3}{2} = 3 \) ways to choose the two boxes designated non-empty. Each fruit can be put into one of those 2 boxes. As a result, there are \( 3 \cdot 2^6 \) ways to put fruit into the boxes such that at least 1 out of 3 boxes is empty.

Finally, using complementary counting to pull it all together, we find there are \( 3^6 - 3 \cdot 2^6 \) ways to arrange the fruits into the boxes such that no box is empty.

Select the correct assertion about the above analysis:
There is nothing wrong with the above analysis.

The analysis over-counts.

The analysis under-counts.
EXPLANATION

The third answer is correct -- we have under-counted. Let \(a, b, c\) be the three distinct boxes. Let \(A, B,\) and \(C\) be the events that boxes \(a, b,\) and \(c\) are empty after we distribute our fruits, respectively.

There are 6 distinct fruits, and each fruit has 3 distinct boxes to put into, so there are \(3^6\) total ways to arrange the fruits into these boxes without any restrictions.

To utilize complementary counting, we want to find the number of ways to arrange the fruits such that at least 1 box out of the 3 is empty. In other words, we need to calculate \(\left|A \cup B \cup C\right|\).

The Inclusion-Exclusion Theorem tells us:

\[
\left|A \cup B \cup C\right| = \left|A\right| + \left|B\right| + \left|C\right| - \left|A \cap B\right| - \left|B \cap C\right| - \left|A \cap C\right| + \left|A \cap B \cap C\right|
\]

Let's break this formula down and interpret it.

\(\left|A\right|, \left|B\right|,\) and \(\left|C\right|\) are the total number of ways to arrange the fruits such that the relevant box is empty. The cardinality of each of these events is the number of ways to put the 6 distinct fruits into the other 2 boxes, i.e., \(2^6\).

\(\left|A \cap B\right|, \left|B \cap C\right|,\) and \(\left|A \cap C\right|\) are the total number of ways to arrange the fruits such that both of the relevant boxes are empty. The number of ways to do this is the number \(\binom{6}{2}\) ways to put the 6 distinct boxes into the one other box: \(\binom{6}{2}\).

\(\left|A \cap B \cap C\right|\) is the total number of ways to arrange the fruits such that all 3 boxes are empty. This is 0 because we have to put the fruits into the boxes, so some box must contain some fruits.

We plug these values back in the formula to get:

\[
\left|A \cup B \cup C\right| = 3 \cdot 2^6 - 3 \cdot 1^6 + 0
= 3 \cdot 2^6 - 3 \cdot 1^6
\]

Finally, we finish our complementary counting, and subtract this value from the unrestricted total. We get:

\[
3^6 - \left(3 \cdot 2^6 - 3 \cdot 1^6\right) = 3^6 - 3 \cdot 2^6 + 3 \cdot 1^6
\]
0 Points

We would love to hear your thoughts, feedback and concerns about this course through this anonymous feedback form: https://forms.gle/pZJ6m2jyRtMsnLNR9

Thanks!