**Q1 Combinations**

1 Point

Anna wants to choose a set of 4 distinct questions for her final exam. She has to select from a test kit consisting of 13 questions. How many different exams can she create if the order of the questions doesn't matter?

(Your answer on this and on the next three problems should be an integer. Remember that you can use Wolfram Alpha to do the calculations for you.)

715

**EXPLANATION**

The test kit contains 13 questions. For the exam, the order of the questions doesn’t matter, and she has to select 4 questions out of 13 possible ones.

Thus, there are \( C'(13, 4) = \binom{13}{4} = \frac{13!}{4!(13-4)!} = 715 \) versions of the final exam.

**Q2 Complementary counting**

1 Point

Luxi wants to bake a cake with 6 layers, each with a unique color. The color of each layer can be 1 of the 6 colors: red, blue, yellow, orange, green, or brown. How many ways can he arrange the layers of the cake so that the top layer is NOT red?

*Hint: Sometimes, it is difficult to directly count the number of objects of interest. One strategy to solve some problems is to switch it around with complementary counting. Instead of trying to count the objects directly, complementary counting involves counting /all/ possible objects and then subtracting from that the number of objects that /don't/ have the property of interest. This will then give you the count of the objects you are interested in since you took away all the ones that don't match your requirements from all possible objects. See how you might try to apply this idea to solve this problem, since it might be difficult to directly count these cakes without red top layers directly. See more 1.1.4 here*
EXPLANATION
Remember the arrangement of the colors is ordered. The total number of ways he can arrange 6 layers of cake with 6 different colors, without any restriction, is: 6!
If the first layer is red, there are 5 layers left for 5 other colors. There are 5! ways to arrange the layers when the first layer is red.
Thus, there are 6! − 5! = 600 ways he can arrange the layers of the cake so that the first layer cannot be red.

Q3 Permutations
1 Point
Hunter has 12 plants and a shelf that is big enough for 5 plants. How many arrangements can the plants possibly have for this small shelf, if order matters? (Each arrangement should contain exactly 5 plants)

95040

EXPLANATION
Shelf arrangement is ordered. We have to arrange 12 plants for 5 slots, so there are \( P(12, 5) = \frac{12!}{(12 - 5)!} = 95040 \) different ways Hunter can arrange the plants on his shelf.

Q4 Anagrams
2 Points
How many ways are there to rearrange the letters in the 12 letter string "AAAAAABBBBCCD"?

=83160+0
EXPLANATION
There are 12 total elements to permute. Of them, 5 are As, 4 are Bs, 2 are Cs, and 1 is a D. We can choose an arrangement by first choosing 5 positions for the As, then among the remaining 7 positions choosing 4 for the Bs, then among the remaining 3 positions choosing 2 for the Cs, and then the D will go in the final spot. This gives

\[(\binom{12}{5}) \cdot (\binom{7}{4}) \cdot (\binom{3}{2}) \cdot 1 = \frac{12!}{5!4!2!1!} = 83,160\]

Q5 Combinatorial Proof
2 Points

We can prove the following equality

\[\sum_{k=0}^{n} \binom{n}{k} = 2^n\]

using the Binomial Theorem. Another approach is to use a combinatorial proof.

To do that, let's say the right hand side counts the number of binary strings of length \(n\) (each character is 0 or 1). Then, what assertions below would show that the left hand side is also equal to this same number.

- ✔️ It counts the number of binary strings of length \(n\), with exactly \(k\) 0’s, summed from \(k = 0\) to \(k = n\).
- ☐ It counts the the number of binary strings of length \(k\), summed from \(k = 0\) to \(k = n\)
- ☐ It counts the the number of binary strings that are equal to \(k\) in binary (e.g., 101 in binary is 5), summed from \(k = 0\) to \(k = n\).
- ✔️ It counts the number of strings of length \(n\), with exactly \(k\) 1’s, summed from \(k = 0\) to \(k = n\).
EXPLANATION

The first and the fourth statements are correct. For this combinatorial argument, we simply need to show that both sides represent the number of binary strings of length $n$.

For the right-hand side, there are two choices for each character in a binary string (0 or 1), so applying the product rule, there are $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ different binary strings of length $n$.

For the left-hand side, we divide the binary strings up into those that have exactly $k$ 0's for each $k$, where $0 \leq k \leq n$ and use the sum rule. The argument is completed by observing that the number of binary strings with exactly $k$ 0's is $\binom{n}{k}$. Thus, the total number of binary strings of length $n$ is: $\sum_{k=0}^{n} \binom{n}{k}$.

Thus, the first assertion completes a combinatorial proof of the fact that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. The proof using the fourth assertion is similar. The remaining assertions are just plain incorrect!