CSE 312: Foundations of Computing II
Quiz Section #6: Important Discrete Random Variables

Review: Main Theorems and Concepts

Variance: Let X be a random variable and $\mu = \mathbb{E}[X]$. The variance of X is defined to be $\text{Var}(X) =$ ______________________________. Notice that since this is an expectation of a ____________ random variable ($(X-\mu)^2$), variance is always ____________. With some algebra, we can simplify this to $\text{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X]$.

Property of Variance: Let $a, b \in \mathbb{R}$ and let X be a random variable. Then, $\text{Var}(aX + b) =$ ____________.

Independence: Random variables X and Y are independent, written $X \perp Y$, iff

____

In this case, we have $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ (the converse is not necessarily true).

i.i.d. (independent and identically distributed): Random variables $X_1, \ldots, X_n$ are i.i.d. (or iid) iff they are ____________ and have the same ______________________________.

Variance of Independent Variables: If $X \perp Y$, $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$. This depends on independence, whereas linearity of expectation always holds. Note that this combined with the above shows that $\forall a, b, c \in \mathbb{R}$ and if $X \perp Y$, $\text{Var}(aX + bY + c) =$ ______________________________.

Zoo of Discrete Random Variables

Uniform: $X \sim Unif(a, b)$, for integers $a \leq b$, iff X has the following probability mass function:

$$p_X(k) = \frac{1}{b - a + 1}, \quad k = a, a+1, \ldots, b$$

$\mathbb{E}[X] = \frac{a+b}{2}$ and $\text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$. This represents each integer from $[a, b]$ to be equally likely. For example, a single roll of a fair die is $Unif(1, 6)$.

Bernoulli (or indicator): $X \sim Ber(p)$ iff X has the following probability mass function:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1 - p, & k = 0 \end{cases}$$

$\mathbb{E}[X] = p$ and $\text{Var}(X) = p(1-p)$. An example of a Bernoulli r.v. is one flip of a coin with $P(\text{head}) = p$. By a clever trick, we can write

$$p_X(k) = p^k (1-p)^{1-k}, \quad k = 0, 1$$

Binomial: $X \sim Bin(n, p)$ iff X is the sum of n iid $Ber(p)$ random variables. X has probability mass function

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

$\mathbb{E}[X] = np$ and $\text{Var}(X) = np(1-p)$. An example of a Binomial r.v. is the number of heads in n independent flips of a coin with $P(\text{head}) = p$. Note that $Bin(1, p) \equiv Ber(p)$. As $n \to \infty$ and $p \to 0$, with $np = \lambda$, then $\text{Bin}(n, p) \to Poi(\lambda)$. If $X_1, \ldots, X_n$ are independent Binomial r.v.'s, where $X_i \sim Bin(N_i, p)$, then $X = X_1 + \ldots + X_n \sim Bin(N_1 + \ldots + N_n, p)$.

Geometric: $X \sim Geo(p)$ iff X has the following probability mass function:

$$p_X(k) = (1-p)^{k-1}\, p, \quad k = 1, 2, \ldots$$

$\mathbb{E}[X] = \frac{1}{p}$ and $\text{Var}(X) = \frac{1-p}{p^2}$. An example of a Geometric r.v. is the number of independent coin flips up to and including the first head, where $P(\text{head}) = p$.

Negative Binomial: $X \sim NegBin(r, p)$ iff X is the sum of r iid $Geo(p)$ random variables. X has probability mass function

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots$$

$\mathbb{E}[X] = \frac{r}{p}$ and $\text{Var}(X) = \frac{r(1-p)}{p^2}$. An example of a Negative Binomial r.v. is the number of independent coin flips up to and including the r^{th} head, where $P(\text{head}) = p$. If $X_1, \ldots, X_n$ are independent Negative Binomial r.v.'s, where $X_i \sim NegBin(r_i, p)$, then $X = X_1 + \ldots + X_n \sim NegBin(r_1 + \ldots + r_n, p)$.

Poisson: $X \sim Poi(\lambda)$ iff X has the following probability mass function:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

$\mathbb{E}[X] = \lambda$ and $\text{Var}(X) = \lambda$. An example of a Poisson r.v. is the number of people born during a particular minute, where λ is the average birth rate per minute. If $X_1, \ldots, X_n$ are independent Poisson r.v.'s, where $X_i \sim Poi(\lambda_i)$, then $X = X_1 + \ldots + X_n \sim Poi(\lambda_1 + \ldots + \lambda_n)$.

Hypergeometric: $X \sim HypGeo(N, K, n)$ iff X has the following probability mass function:

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad k = max\{0, n+K-N\}, \ldots, \min\{K, n\}$$

$\mathbb{E}[X] = n\frac{K}{N}$. This represents the number of successes drawn, when n items are drawn from a bag with N items (K of which are successes, and $N-K$ failures) without replacement. If we did this with replacement, then this scenario would be represented as $\text{Bin}\left(n, \frac{K}{N}\right)$.

Exercises

1. Suppose I am fishing in a pond with B blue fish, R red fish, and G green fish, where $B + R + G = N$. For each of the following scenarios, identify the most appropriate distribution (with parameter(s)):

   (a) how many of the next 10 fish I catch are blue, if I catch and release

   (b) how many fish I had to catch until my first green fish, if I catch and release

   (c) how many red fish I catch in the next five minutes, if I catch on average r red fish per minute

   (d) whether or not my next fish is blue

   (e) how many of the next 10 fish I catch are blue, if I do not release the fish back to the pond after each catch

   (f) how many fish I have to catch until I catch three red fish, if I catch and release

2. Suppose $Y_1, \ldots, Y_n$ are iid with $\mathbb{E}[Y_i] = \mu$ and $\text{Var}(Y_i) = \sigma^2$, and let $Y = \frac{1}{n}\sum_{i=1}^n iY_i$. What is $\mathbb{E}[Y]$ and $\text{Var}(Y)$? Recall that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

3. Is the following statement true or false? If $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, then $X \perp Y$. If it is true, prove it. If not, provide a counterexample.

4. Suppose we roll two fair 5-sided dice independently. Let X be the value of the first die, Y be the value of the second die, $Z = X + Y$ be their sum, $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$.

   (a) Find $p_U(u)$.

   (b) Find $\mathbb{E}[U]$.

   (c) Find $\mathbb{E}[Z]$.

   (d) Find $\mathbb{E}[UV]$.

   (e) Find $Var(U + V)$.

5. Suppose X has the following probability mass function:

$$p_X(x) = \begin{cases} c, & x = 0 \\ 2c, & x = \frac{\pi}{2} \\ c, & x = \pi \\ 0, & \text{otherwise} \end{cases}$$

   (a) Suppose $Y_1 = \sin(X)$. Find $\mathbb{E}[Y_1^2]$.

   (b) Suppose $Y_2 = \cos(X)$. Find $\mathbb{E}[Y_2^2]$.

   (c) Suppose $Y = Y_1^2 + Y_2^2 = \sin^2(X) + \cos^2(X)$. Before any calculation, what do you think $\mathbb{E}[Y]$ should be? Find $\mathbb{E}[Y]$, and see if your hypothesis was correct. (Recall for any real number x, $\sin^2(x) + \cos^2(x) = 1$).

   (d) Let W be any discrete random variable with probability mass function $p_W(w)$. Then, $\mathbb{E}[\sin^2(W) + \cos^2(W)] = 1$. Is this statement always true? If so, prove it. If not, give a counterexample by giving a probability mass function for a discrete random variable W for which the statement is false.

6. If electricity power failures occur according to a Poisson distribution with an average of 3 failures every twenty weeks, calculate the probability that there will be more than one failure during a particular week.

7. An average page in a book contains one typo. What is the probability that there are exactly 8 typos in a given 10-page chapter, using the Poisson model?

8. A company makes electric motors. The probability an electric motor is defective is 0.01, independent of other motors made. What is the probability that a sample of 300 electric motors will contain exactly 5 defective motors? Do it first exactly, then approximate it with a Poisson distribution. How good is the approximation?

9. Suppose that $X_1 \sim Poi(\lambda_1)$ and $X_2 \sim Poi(\lambda_2)$, and X_1 and X_2 are independent. Prove that $Y = X_1 + X_2 \sim Poi(\lambda_1 + \lambda_2)$.