CSE 312: Foundations of Computing II

Quiz Section #2: Inclusion-Exclusion, Pigeonhole, Introduction to Probability

Review: Main Theorems and Concepts

Binomial Theorem: ________________________________

Principle of Inclusion-Exclusion (PIE): 2 events: $|A \cup B| =$ ______________________
3 events: $|A \cup B \cup C| =$ ______________________
In general: ______________________

Pigeonhole Principle: If there are n pigeons with k holes and $n > k$, then at least one hole contains at least ______________________ pigeons.

Complementary Counting (Complementing): If asked to find the number of ways to do X, you can: ________________________________

Sample Space: The set of all possible outcomes of an experiment, denoted Ω or S
Event: Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
Union: The union of two events E and F is denoted $E \cup F$
Intersection: The intersection of two events E and F is denoted $E \cap F$ or EF
Mutually Exclusive: Events E and F are mutually exclusive iff $E \cap F = \emptyset$
Complement: The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \setminus E$
DeMorgan's Laws: $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
Probability of an event E: denoted $\mathbb{P}(E)$ or $\Pr(E)$ or $P(E)$
Partition: Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff

- $E_1, \ldots, E_n$ are exhaustive: ______________________, and
- $E_1, \ldots, E_n$ are pairwise mutually exclusive: ______________________
  - Note that for any event A (with $A \neq \emptyset, A \neq \Omega$): ________ and ________ partition Ω

Axioms of Probability and their Consequences

1. **(Non-negativity)** For any event E, $\mathbb{P}(E) \geq$ ________
2. **(Normalization)** $\mathbb{P}(\Omega) =$ ________
3. **(Additivity)** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) =$ ______________________

Corollaries of these axioms:

- $\mathbb{P}(E) + \mathbb{P}(E^C) =$ ________

- If $E \subseteq F$, $\mathbb{P}(E)$ ________ $\mathbb{P}(F)$
- $\mathbb{P}(E \cup F) =$ ___________________________

Equally Likely Outcomes: If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) =$___________________________.

- Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

Conditional Probability: $\mathbb{P}(A|B) =$ ___________________________

Exercises

1. Give a **combinatorial** proof that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Do not use the binomial theorem. (Hint: you can count the number of subsets of $[n] = \{1, 2, \ldots, n\}$). Note: A combinatorial proof is one in which you explain how to count something in two different ways – then those formulae must be equivalent if they both indeed count the same thing.

2. How many ways are there to choose three initials (upper case letters) such that two are the same or all three are the same?

3. Suppose there are N items in a bag, with K of them marked as successes in total (and the rest are marked as failures). We draw n of them, without replacement. Each item is equally likely to be drawn. Let X be the number of successes we draw (out of n). What is $\mathbb{P}(X = k)$, that is, the probability we draw exactly k successes?

4. Suppose we have 12 chairs (in a row) with 9 TA's, and Professors Ruzzo, Karlin, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to his/her immediate left and right?

5. Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. Each action is specified by saying whether he puts on a sock or a shoe, and saying which leg he puts it on. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and all shoes are indistinguishable from each other.

6. Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent. Repeat the question for the letters "AAAAABBB".

7. Given 3 different spades and 3 different hearts, shuffle them. Compute $\mathbb{P}(E)$, where E is the event that the suits of the shuffled cards are in alternating order. What is your sample space?

8. Suppose you pick two cards from a well-shuffled Schnapsen deck. What is the probability that they are both queens?

9. At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). N people each pick 2 cards from the deck and hold onto them. What is the minimum value of N that guarantees at least 2 people have the same combination of suits?

10. Suppose you deal 13 cards from a well-shuffled bridge deck (4 suits with 13 cards in each). What is the probability that the distribution of suits is 4, 4, 3, 2? (That is, you have 4 cards of one suit, 4 cards of another suit, 3 cards of another suit, and 2 cards of the last suit.)

11. Novice poker players are often confused about which player wins if one holds a flush and one holds a straight. For draw poker (see quiz section #1 worksheet, exercise #25):

    (a) Compute the probability of being dealt a flush.

    (b) Compute the probability of being dealt a straight.

    (c) Which of these hands should win, given your answers to (a) and (b)?

12. This is another poker exercise. Find the minimum number of cards to be dealt to you from a standard 52-card deck to guarantee that you have some 5 cards among them that form ...

    (a) one pair? (This occurs when the cards have ranks a, a, b, c, d, where a, b, c, and d are all distinct. The suits do not matter.)

    (b) two pairs? (This occurs when the cards have ranks a, a, b, b, c, where a, b, and c are all distinct. The suits do not matter.)

    (c) a full house? (This occurs when the cards have ranks a, a, a, b, b, where a and b are distinct. The suits do not matter.)

    (d) a straight? (A hand is said to form a straight if the ranks of all 5 cards form an incrementing sequence. The suits do not matter. The lowest straight is A, 2, 3, 4, 5 and the highest straight is 10, J, Q, K, A.)

    (e) a flush? (A hand is said to form a flush if all 5 cards are from the same suit.)

    (f) a straight flush (5 cards of the same suit that form a straight)?

13. In Schnapsen, suppose that ♠J is the face-up trump and you are dealt 5 nontrump cards. Let E be the event that the top 4 cards in the stock are all trumps. Let the sample space be all possible orderings of all the cards in the stock. Compute $P(E)$. (Notice that your solution suggests a different and simpler sample space.)

14. Suppose you are taking a multiple-choice test that has c answer choices for each question. In answering a question on this test, the probability that you know the correct answer is p. If you don't know

the answer, you choose one at random, with each choice equally probable. What is the probability that you knew the correct answer to a question, given that you answered it correctly?

15. An urn contains 3 black balls and 4 white balls.

    (a) Suppose 3 balls are drawn from the urn without replacement. What is the probability that all 3 are white? Try computing this in the sample space where the order of the 3 draws does not matter, and then in the sample space where the order does matter.

    (b) Suppose 3 balls are drawn from the urn with replacement. What is the probability that all 3 are white? Describe the sample space precisely.

16. At a dinner party, the n people present are to be seated uniformly spaced around a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down at the correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

17. (a) Two parents only have 3 bedrooms for their 13 children. If each child is assigned to a bedroom, one of the bedrooms must have at least c children. What is the maximum value of c that makes this statement true? Prove it.

    (b) (Strong Pigeonhole Principle) More generally, what can you say about n children in k bedrooms? Find a general formula for the maximum value of c that guarantees one of the bedrooms must have at least c children.

18. Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.

    (a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

    (b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 9 Republicans and 1 Democrat?

19. A couple has 2 children. What is the probability that both are girls, given that the older one is a girl?

20. What is the probability that at least one of a pair of fair dice comes up 5, given that the sum of the dice is 8?

21. A plane has 100 seats and 100 passengers. The first person to get on the plane lost his ticket and doesn't know his assigned seat, so he picks a random seat to sit in, with each seat equally probable. Every remaining person knows their seat, so if it is available they sit in it, and if it is unavailable they pick a random remaining seat, with each unoccupied seat equally probable. What is the probability the last person to get on gets to sit in his own seat?

22. (The "Monty Hall" puzzle) Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host,

who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?

23. (Challenge problem) n people at a reception give their hats to a hat-check person. When they leave, the hat-check person gives each of them a hat chosen at random. What is the probability that no one gets their own hat back?