Review: Main Theorems and Concepts

1. **Product Rule**: Suppose there are \( m_1 \) possible outcomes for event \( A_1 \), then \( m_2 \) possible outcomes for event \( A_2 \), ..., \( m_n \) possible outcomes for event \( A_n \). Then there are \( m_1 \cdot m_2 \cdot m_3 \cdots m_n = \prod_{i=1}^{n} m_i \) possible outcomes overall.

2. **Number of ways to order \( n \) distinct objects**: \( n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1 \)

3. **Number of ways to select from \( n \) distinct objects**:
   
   (a) **Permutations** (number of ways to linearly arrange \( k \) objects out of \( n \) distinct objects, when the order of the \( k \) objects matters):
   
   \[
P(n, k) = \frac{n!}{(n-k)!}
   \]

   (b) **Combinations** (number of ways to choose \( k \) objects out of \( n \) distinct objects, when the order of the \( k \) objects does not matter):

   \[
   \frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n, k)
   \]

4. **Multinomial coefficients**: Suppose there are \( n \) objects, but only \( k \) are distinct, with \( k \leq n \). (For example, “godoggy” has \( n = 7 \) objects (characters) but only \( k = 4 \) are distinct: \((g, o, d, y))\). Let \( n_i \) be the number of times object \( i \) appears, for \( i \in \{1, 2, \ldots, k\} \). (For example, \((3, 2, 1, 1)\), continuing the “godoggy” example.) The number of distinct ways to arrange the \( n \) objects is:

   \[
   \frac{n!}{n_1!n_2!\cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}
   \]

**Exercises**

Several exercises below deal with a “standard” 52-card deck, such as is used in the games of bridge and poker. This deck consists of 52 cards divided into 4 suits of 13 cards each. The 4 suits are
(black) spades ♠, (red) hearts ♥, (black) clubs ♣, and (red) diamonds ♦. The 13 cards (“ranks”) of each suit are 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A.

1. How many ways are there to select 5 cards from a standard deck of 52 cards, where the 5 cards contain cards from at most two suits, if:

(a) order does not matter

Case 1: all from the same suit. Choose 1 of 4 suits, and 5 cards from that suit

\[
\binom{4}{1} \binom{13}{5}
\]

Case 2: from two suits. Choose 2 of 4 suits: 1 from the first and 4 from the second, 2 from the first and 3 from the second, etc.)

\[
\binom{4}{2} \cdot \left[ \binom{13}{1} \binom{13}{4} + \binom{13}{2} \binom{13}{3} + \binom{13}{3} \binom{13}{2} + \binom{13}{4} \binom{13}{1} \right]
\]

Our total is

\[
\binom{4}{1} \binom{13}{5} + \binom{4}{2} \cdot \left[ \binom{13}{1} \binom{13}{4} + \binom{13}{2} \binom{13}{3} + \binom{13}{3} \binom{13}{2} + \binom{13}{4} \binom{13}{1} \right]
\]

Let’s talk about an incorrect solution:
Step 1: First choose the two suits from which the cards will come: \(\binom{4}{2}\) possibilities
Step 2: Then choose the 5 cards from among the 26 possible cards of those suits: \(\binom{26}{5}\)
Thus, the total number is ways is \(\binom{4}{2}\binom{26}{5}\)

Why this is wrong:

The problem is that this method overcounts some choices. In particular, a choice consisting of cards that are entirely from one suit, say hearts, will be counted 3 times:

- once when the two suits selected are Hearts/Spades,
- once when the two suits selected are Hearts/Diamonds, and
- once when the two suits selected are Hearts/Clubs

Applying “The Sleuth Principle”, given an outcome selected according to some application of the product rule, we need to be able to reconstruct exactly what choice was made at each step, or else we have made a mistake. When we see an outcome consisting of all hearts, we cannot reconstruct the choice made in the first step: it could have been any of the 3 possibilities mentioned above.

To correct this, one can subtract off the overcounted stuff which is
(b) order matters

Just 5! times the previous answer, since we can permute the 5 distinct cards that many ways. \[ 5! \cdot \left( \binom{4}{1} \binom{13}{5} + \binom{4}{2} \cdot \left[ \binom{13}{1} \binom{13}{4} + \binom{13}{2} \binom{13}{3} + \binom{13}{3} \binom{13}{2} + \binom{13}{4} \binom{13}{1} \right] \right) \]

2. Consider a set of 25 people that form a social network. The structure of the social network is determined by which pairs of people in the group are “friends”. How many possibilities are there for the structure of this social network?

There are \( \binom{25}{2} \) possible undirected edges representing friendships, and each is either there or not, so the number is \( 2^{\binom{25}{2}} \)

3. Suppose we have 3 diamonds and 3 hearts from a standard deck. How many ways are there to arrange the cards if they have to alternate suit?

Method 1: 6 possible cards for the first location, then 3 because you can’t choose the same suit. Then 2 for the third location, because the suit is determined by the first location and there are only 2 cards left in that suit. Similarly 2 for the fourth location. Then 1 choice for each of the fifth and sixth locations. \( 6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \)

Method 2: Find the arrangements individually for each of the suits: 3! for each suit. They have to be alternating, but there are 2 choices for which suit comes first, and then the order will be determined. \( 2 \cdot (3!)^2 \)

Check that the answers are equivalent.

4. How many ways are there to choose three initials that have two being the same or all three being the same?

Complementary counting. Count the total \( 26^3 \) and subtract the number with all distinct initials \( 26 \cdot 25 \cdot 24 = P(26, 3) \) to get \( 26^3 - P(26, 3) \).

5. A license plate has the form AXYZBCD, where A, B, C, and D are digits and X, Y, and Z are upper case letters. What is the number of different license plates that can be created?

\( 10^4 \cdot 26^3 = 175,760,000 \)
6. A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert combinations are there for the week?

Start from Thursday and work forward and backward in the week: \(4 \cdot 4 \cdot 4 \cdot 1 \cdot 4 = 4^6 = 4096\)

7. A store has 4 books, 14 movies, 6 toys, and 5 posters. In how many ways can a customer buy exactly 1 item from each of exactly 3 categories?

\[4 \cdot 14 \cdot 6 + 4 \cdot 14 \cdot 5 + 4 \cdot 6 \cdot 5 + 14 \cdot 6 \cdot 5 = 1156\]

8. In Schnapsen, assuming the stock is not closed, no one has exchanged the jack of trumps, and no marriage has been declared, how many possible orderings of the cards face-down in the stock are there, given the cards you have seen . . .

(a) . . . before trick 1? The number of 9-permutations of the 14 unseen cards, \(P(14, 9) = \frac{14!}{(14-9)!} = \frac{14!}{5!} = 726,485,760\)

(b) . . . before trick 2? \(\frac{12!}{5!} = 3,991,680\)

(c) . . . before trick 3? \(\frac{10!}{5!} = 30,240\)

(d) . . . before trick 4? \(\frac{8!}{5!} = 336\)

(e) . . . before trick 5? \(\frac{6!}{5!} = 6\)

9. In how many different ways can you arrange seven people around a circular table?

\(7!/7 = 6! = 720.\) In general for \(n\) objects arranged in a circle, the answer is \(n!/n = (n - 1)!;\)
if you imagine the \(n!\) permutations of the objects in a linear sequence, this counts each of the circular arrangements \(n\) times, because there are \(n\) different places you can “cut” the circle to get a different linear arrangement.

10. Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

\[(2 \cdot 7 + 2 \cdot 6)6! = 18720\]

11. Maestro Tompa and 6 TAs line up for a picture. How many possible arrangements are there with Maestro Tompa not at either end of the line?
12. How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

\[ 8! - 2 \cdot 7! + 6! = 30960 \]

13. There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

\[ (10!/5!)35! \]

14. Permutations of objects, some of which are indistinguishable.

(a) How many permutations are there of the letters in DAWGY? \[ 5! = 120 \]

(b) How many permutations are there of the letters in DOGGY? \[ 5!/2! = 60 \]

(c) How many permutations are there of the letters in GODOGGY? \[ 7!/(3!2!1!1!) = 420 \]

15. A bridge hand consists of 13 cards dealt from a shuffled standard deck of 52 cards. Given a bridge hand consisting of 5 spades, 2 hearts, 3 diamonds, and 3 clubs, in how many ways can the hand be arranged so that the cards of each suit are together . . .

(a) . . . but not necessarily sorted by rank within each suit? \[ 4!5!2!3!/3! = 207360 \]

- 4! ways to order all the suits
- 5! ways to order the spades
- 2! ways to order the hearts
- 3! ways to order the diamonds
- 3! ways to order the clubs

(b) . . . and each suit is sorted in ascending rank order? \[ 4! = 24 \]

(c) . . . and each suit is sorted in ascending rank order and the suits are arranged so that the suit colors alternate? \[ 4 \cdot 2 \cdot 1 \cdot 1 = 8 \] (4 options for what suit is first, 2 options for the next suit because it has to be of the other color, then one option each for the remaining two suits)
16. Suppose two cards are drawn in order from a bridge deck. In how many ways can the first card be a diamond and the second card a jack?

\[ 13 \cdot 4 - 1 = 51 \]

17. Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

If Peter and Pauline go to the same store, there are 4 stores it could be. For each such choice, there are 3 choices of store for each of the 3 offspring, so \( 3^3 \) choices for all the offspring. If Peter and Pauline go to different stores, there are \( 4 \cdot 3 = 12 \) pairs of stores they could go to. For each such choice, there are 2 choices of store for each of the 3 offspring, so \( 2^3 \) choices for all the offspring. Therefore the answer is \( 4 \cdot 3^3 + 12 \cdot 2^3 = 204 \).

18. You are playing a game of Schnapsen against the Maestro. The cards you have not seen yet during the current deal are the following:

\begin{itemize}
  \item ♠ TKJ
  \item ♥ ATQJ
  \item ♠ AT
  \item ♦ KQJ
\end{itemize}

Of the possible 5-card hands the Maestro could be holding, how many of them contain at least 18 trick points? Try to find the simplest way to solve this exercise.

Any hand that contains an ace or ten has at least as many trick points as TQJJJ, which is 19. Any hand that does not contain an ace or ten has at most as many trick points as KKQQJ, which is 16. Since there are 12 unseen cards and 5 aces and tens, the answer is \( \binom{12}{5} - \binom{12-5}{5} = 771 \).

19. You have 12 red beads, 16 green beads, and 20 blue beads. How many distinguishable ways are there to place the beads on a string, assuming that beads of the same color are indistinguishable? (The string has a loose end and a tied end, so that reversing the order of the beads gives a different arrangement, unless the pattern of colors happens to form a palindrome.) Try solving the problem two different ways, once using permutations and once using using combinations.

Using permutations:

\[ \frac{48!}{12! \cdot 16! \cdot 20!} \]
Using combinations:
\[
\binom{48}{12} \binom{36}{16} = \frac{48!}{12! \cdot 36!} \cdot \frac{36!}{16! \cdot 20!}
\]

20. There are 12 points on a plane. Five of them are collinear and, other than these, no three are collinear.

(a) How many lines, each containing at least 2 of the 12 points, can be formed?
\[
\binom{12}{2} - \binom{5}{2} + 1 = 57
\]

(b) How many triangles, each containing at least 3 of the 12 points, can be formed?
\[
\binom{12}{3} - \binom{5}{3} = 210
\]

21. You have a triangular prism with top and bottom both being congruent equilateral triangles and the three sides being congruent rectangles. If you pick 5 out of 7 different colors, one to paint each of the 5 faces, how many differently painted triangular prisms can you get? Just rotating the prism does not constitute a different color scheme.
\[
\frac{7!}{2! \cdot 2 \cdot 3} = 420
\]

There are 2 \cdot 3 rotations of the prism that leave the 5 faces in their original positions. That means that \(P(7, 5)\) counts each color scheme 6 times.

22. There are 6 men and 7 women in a ballroom dancing class. If 4 men and 4 women are chosen and paired off, how many pairings are possible?
\[
\binom{6}{4} \binom{7}{4} 4! = 12,600
\]

23. How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if...

(a) ...the seats are assigned arbitrarily?
\[
10!
\]
(b) . . . all couples are to get adjacent seats?

\[
10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 2^5 \cdot 5!\text{.}
\]

there are 5! permutations of the 5 couples, and then 2 permutations within each of the 5 couples.

(c) . . . the seats are assigned arbitrarily, except that one couple insists on not sitting in adjacent seats?

There are 9! \cdot 2 arrangements in which this couple does sit in adjacent seats, since you can treat the couple as a ninth unit added to the other 8 individuals, and then there are 2 permutations of that couple’s seats. That means the answer to the question is

\[
10! - 9! \cdot 2 = 8 \cdot 9!.
\]

24. How many bridge hands have a suit distribution of 5, 5, 2, 1? (That is, you are playing with a standard 52-card deck and you have 5 cards of one suit, 5 cards of another suit, 2 of another suit, and 1 of the last suit.)

\[
\binom{13}{5}\binom{13}{5}\binom{13}{2}\binom{13}{1} \cdot \frac{4!}{2!}\text{.}
\]

the factor of 4! in the numerator takes care of the number of ways to assign suits to the number of cards, and the factor of 2! in the denominator takes care of the fact that two suits have the same number (5) of cards and so are overcounted.

25. A hand in “draw poker” consists of 5 cards dealt from a shuffled 52-card standard deck.

(a) How many different hands are there that form a flush? (A hand is said to form a flush if all 5 cards are from the same suit.)

\[
\binom{4}{13} \binom{13}{5}\text{.}
\]

4 choices of the suit and \( \binom{13}{5} \) choices of 5 cards from that suit

(b) How many different hands are there that form a straight? (A hand is said to form a straight if the ranks of all 5 cards form an incrementing sequence. The suits do not matter. The lowest straight is A, 2, 3, 4, 5 and the highest straight is 10, J, Q, K, A.)

\[
10 \cdot 4^5\text{.}
\]

10 choices for the rank of the lowest card in the straight and then 4 choices for the suit of each of the 5 cards.

(c) How many different hands are there that form one pair? (This occurs when the cards have ranks a, a, b, c, d, where a, b, c, and d are all distinct. The suits do not matter.)

\[
13\binom{4}{2} \binom{12}{3} \cdot 4^3\text{.}
\]

13 choices for the value of a, \( \binom{4}{2} \) choices for the suits of the 2 cards
of rank $a$, $\binom{12}{3}$ choices for \{b, c, d\}, and 4 choices for the suit of each of these 3 cards

(d) How many different hands are there that form two pairs? (This occurs when the cards have ranks $a$, $a$, $b$, $b$, $c$, where $a$, $b$, and $c$ are all distinct. The suits do not matter.)

$$\binom{13}{2}\left(\frac{4}{2}\right)^2 \cdot 11 \cdot 4 \cdot \binom{13}{2} \text{ choices for } \{a, b\} \text{ and } 11 \text{ choices for the value of c.}$$

(e) How many different hands are there that form three of a kind? (This occurs when the cards have ranks $a$, $a$, $a$, $b$, $c$, where $a$, $b$, and $c$ are all distinct. The suits do not matter.)

$$13\left(\frac{4}{3}\right) \cdot \binom{12}{2} \cdot 4^2$$

(f) How many different hands are there that form a full house? (This occurs when the cards have ranks $a$, $a$, $a$, $b$, $b$, where $a$ and $b$ are distinct. The suits do not matter.)

$$13\left(\frac{4}{3}\right) \cdot 12 \cdot \left(\frac{4}{2}\right)$$

(g) How many different hands are there that form four of a kind? (This occurs when the cards have ranks $a$, $a$, $a$, $a$, $b$. The suits do not matter.)

$$13 \cdot 12 \cdot 4$$

(h) How many different hands are there that form a straight flush? (This occurs when the cards form a straight and a flush; i.e., a straight with all 5 cards of the same suit)

$$10 \cdot 4 = 40$$. We have 10 possible straights as in part (b), but we have to choose one of 4 suits for all the cards.