Proof: Let $X = (Y-\mu)^2$. X is nonnegative, so

$$P(|Y-\mu| \geq \alpha) = P((Y-\mu)^2 \geq \alpha^2)$$
$$= P(X \geq \alpha^2) \leq E[X]/\alpha^2 \quad \text{(Markov)}$$
$$= E[(Y-\mu)^2]/\alpha^2 = Var(Y)/\alpha^2$$

Ex: If Y is daily business cost, with $E[Y] = 1500 = \mu$ and $\sigma = \sqrt{Var(Y)} = 200$.

$$P(Y \geq 2500) = P(Y-\mu \geq 1000) \leq P(|Y-\mu| \geq 1000)$$
$$\leq \frac{Var(Y)}{1000^2} = \left(\frac{200}{1000}\right)^2 = \frac{1}{25}$$

Cantelli's Inequality: (one-sided Chebyshev)

$$P(Y-\mu \geq \alpha) \leq \frac{Var(Y)}{Var(Y)+\alpha^2}$$

Ex: Same example:

$$P(Y \geq 2500) = P(Y-\mu \geq 1000) \leq \frac{200^2}{200^2+1000^2}$$
$$= \frac{1}{1+\left(\frac{1000}{200}\right)^2} = \frac{1}{26}$$

Chernoff Bound:

Theorem: Suppose $X \sim Bin(n,p)$ and $\mu = E[X]$. For any $0 < \delta < 1$,

$$P(X \geq (1+\delta)\mu) \leq e^{-\frac{1}{3}\delta^2\mu} \quad \text{and}$$
$$P(X \leq (1-\delta)\mu) \leq e^{-\frac{1}{2}\delta^2\mu}.$$

Ex: $X \sim Bin(10, 1/2)$, $\delta = 0.4$, $\mu = 10 \cdot \frac{1}{2} = 5$.

$$P(X \geq 1.4\mu) = P(X \geq 7) \leq e^{-\frac{1}{3}(0.4)^2 5}$$

Law of Large Numbers

Consider iid r.v.'s $X_1, X_2, X_3, \ldots$, where $E[X_i] = \mu < \infty$ and $Var(X_i) = \sigma^2 < \infty$.

Define sample mean $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$$E[\overline{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu$$

$$= \frac{1}{n}\cdot n\mu = \mu.$$

$$Var(\overline{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

As n increases, $\overline{X}_n$ is more likely to be close to μ.

Theorem (Weak Law of Large Numbers):

For any $\varepsilon > 0$, as $n \to \infty$,

$$P(|\overline{X}_n - \mu| > \varepsilon) \to 0.$$

Proof: By Chebyshev's inequality,

$$P(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty$$