Example of Central Limit Theorem
Worksheet #8, Exercise #6.

Let X be the number of crash-free days.
Then $X \sim Bin(100, 0.9)$.

$E[X] = 100 \times 0.9 = 90$,

$Var(X) = 100 \times 0.9(1-0.9) = 9$

$$P(X \geq 87) = P(87 \leq X \leq 100) = P(86.5 < X < 100.5)$$

$$= P\left(\frac{86.5-90}{3} < \frac{X-90}{3} < \frac{100.5-90}{3}\right)$$

$$\approx P\left(-1.17 < \frac{X-90}{3} < 3.5\right)$$

$$\approx \Phi(3.5) - \Phi(-1.17)$$

(Central Limit Theorem approximates X)

$$= \Phi(3.5) - (1 - \Phi(1.17))$$

$$= \Phi(3.5) + \Phi(1.17) - 1$$

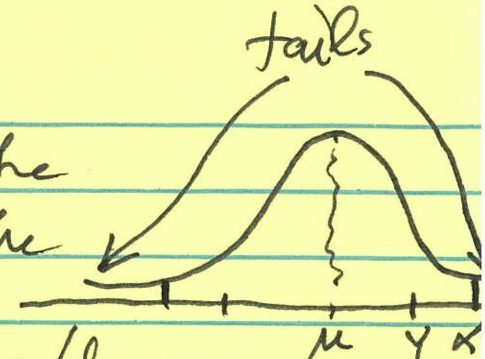
$$\approx 0.9998 + 0.8790 - 1$$

$$= 0.8788$$

$$P(X \geq 87) = P(X > 86.5) \approx P\left(\frac{X-90}{3} > -1.17\right)$$

$$\approx \Phi(1.17) = 0.8790$$

The extremes $X < 0$ or $X > n$ can always be ignored if $np(1-p) \geq 10$, when $X \sim Bin(n,p)$.



Tail Bounds

Useful to be able to bound the probability of being far from the mean.

Ex: Expected business costs are \$1500/day. What is the probability that it's \$6000 on one particular day?

Markov's ~~Equality~~ Inequality

Theorem: If X is a nonnegative random variable, then for any $\alpha > 0$,

$$P(X \geq \alpha) \leq E[X]/\alpha$$

Ex: $P(X \geq 6000) \leq E[X]/6000 = 1500/6000 = 1/4$

Proof: $E[X] = \sum_x x\,p_X(x) = \sum_{x<\alpha} x\,p_X(x) + \sum_{x\geq\alpha} x\,p_X(x)$

$$\geq 0 + \alpha \sum_{x\geq\alpha} p_X(x) = \alpha P(X \geq \alpha)$$

$$P(X \geq \alpha) \leq E[X]/\alpha$$

Corollary: $P(X \geq k\,E[X]) \leq 1/k$ for any $k > 0$, if X is always nonnegative.

Chebyshev's Inequality

Theorem: If Y is a random variable with $E[Y] = \mu$, then for any $\alpha > 0$,

$$P(|Y - \mu| \geq \alpha) \leq \frac{Var(Y)}{\alpha^2}.$$

Corollary: Let $\mu = E[Y]$ and $\sigma = \sqrt{Var(Y)}$. Then for any $k > 0$, $P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}$