Example of Central Limit Theorem

Worksheet #8, Exercise #6.

Let $X$ be the number of crash-free days. Then $X \sim \text{Bin}(100, 0.9)$.

$\text{E}[X] = 100 \times 0.9 = 90.$

$\text{Var}(X) = 100 \times 0.9 \times (1 - 0.9) = 9.$

$P(X \geq 87) = P(87 \leq X \leq 100) = P(86.5 < X < 100.5)$

$= P\left(\frac{86.5 - 90}{3} < \frac{X - 90}{3} < \frac{100.5 - 90}{3}\right)$

$\approx P(-1.17 < \frac{X - 90}{3} < 3.5)$

$\approx \Phi(3.5) - \Phi(-1.17)$ (Central Limit Theorem approximates $X$)

$= \Phi(3.5) - (1 - \Phi(1.17))$

$= \Phi(3.5) + \Phi(1.17) - 1$

$\approx 0.9998 + 0.8790 - 1$

$= 0.8788$

$P(X \geq 87) = P(X > 86.5) \approx P\left(\frac{X - 90}{3} > -1.17\right)$

$\approx \Phi(1.17) = 0.8790$

The extremes $X < 0$ or $X > n$ can always be ignored if $np(1-p) \geq 10$, when $X \sim \text{Bin}(n, p)$. 
Tail Bounds
Useful to be able to bound the probability of being far from the mean.

Ex: Expected business costs are $1500/day.
What is the probability that it is $6000 on one particular day?

Markov's Inequality Inequality
Theorem: If $X$ is a nonnegative random variable, then for any $\alpha > 0$,
$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

Ex: $P(X \geq 6000) \leq \frac{E[X]}{6000} = \frac{\frac{1500}{6000} = \frac{1}{4}}{6000}$

Proof: $E[X] = \sum_{x < \alpha} xP(x) = \sum_{x < \alpha} xP(x) + \sum_{x \geq \alpha} xP(x)$

$\geq 0 + \alpha \sum_{x \geq \alpha} P(x) = \alpha P(X \geq \alpha)$

$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$

Corollary: $P(X \geq kE[X]) \leq \frac{1}{k}$ for any $k > 0$, if $X$ is always nonnegative.

Chebyshev's Inequality
Theorem: If $Y$ is a random variable with $E[Y] = \mu$, then for any $\alpha > 0$,
$$P(|Y - \mu| \geq \alpha) \leq \frac{\text{Var}(Y)}{\alpha^2}$$

Corollary: Let $\mu = E[Y]$ and $\sigma = \sqrt{\text{Var}(Y)}$. Then for any $k \geq 0$, $P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}$