Continuous Random Variables

A continuous r.v. takes values from an uncountably infinite set. For us, $\mathbb{R}$ = real numbers.

Ex: Height of a randomly chosen person.
Waiting time until arrival of next packet.

Defn: For a r.v. X, the cumulative distribution function (cdf) is defined as

$$F_X(x) = P(X \le x).$$

Monotonically nondecreasing: $x < y \Rightarrow F_X(x) \le F_X(y)$.

Defn: A continuous random variable X is one whose cdf $F_X(x): \mathbb{R} \to \mathbb{R}$ is continuous everywhere

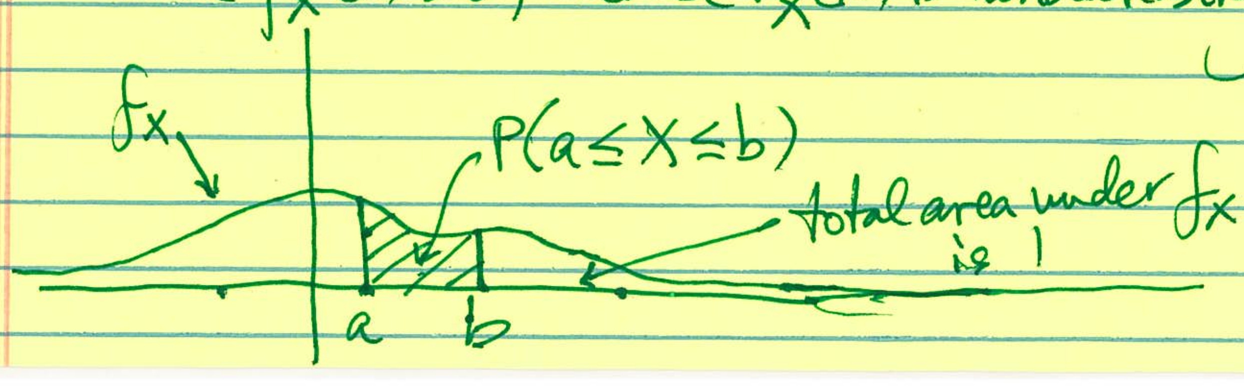
Defn: Let X be a continuous r.v. Then the probability density function (density, pdf) is $f_X(x): \mathbb{R} \to \mathbb{R}$, defined as $f_X(x) = \frac{d}{dx} F_X(x)$.

Turning this around, $F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$

It follows $P(a \le X \le b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx$

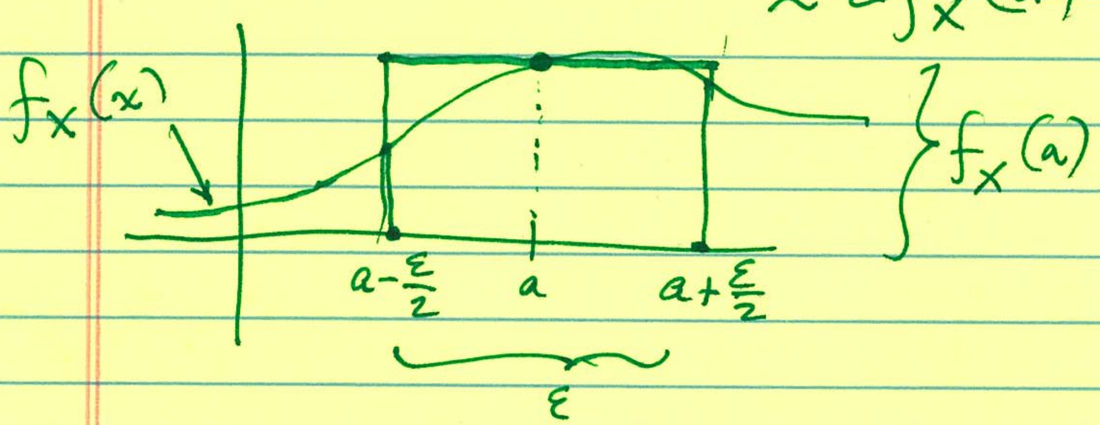
and $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

and $f_X(x) \ge 0$, because $F_X(x)$ is nondecreasing.



Remarks:

1. Densities are not probabilities: they may be > 1.
2. $P(X=a) = P(a \le X \le a) = F_X(a) - F_X(a) = 0$.
3. $P(a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}) = F_X(a + \frac{\varepsilon}{2}) - F_X(a - \frac{\varepsilon}{2})$
$\approx \varepsilon f_X(a)$



so $P(X$ is "near" $a)$ is proportional to $f_X(a)$.

For continuous r.v.'s X, usually substitute $\int$ for Σ and f_X for p_X.

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$
$$E[aX+b] = aE[X]+b$$
$$E[X+Y] = E[X]+E[Y].$$
$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$$
$$Var(X) = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

X and Y are independent iff
$$\forall A \forall B\ P(X \in A \cap Y \in B) = P(X \in A)P(Y \in B).$$