Linearity: $E[X+Y] = E[X] + E[Y]$ is special.
In general, $E[XY] \neq E[X]E[Y]$

$$E[X^2] \neq (E[X])^2$$
$$E[\sqrt{X}] \neq \sqrt{E[X]}$$

etc

Variance: Consider two fair coin games between A and B:

1. A's gain per flip is $X = \begin{cases} +1, & \text{if heads} \\ -1, & \text{if tails} \end{cases}$
2. A's gain per flip is $Y = \begin{cases} +1000, & \text{if heads} \\ -1000, & \text{if tails} \end{cases}$

$E[X] = E[Y] = 0$, so are you equally happy to play either game?

$E[X]$ measures the average value of X. What about the variability of X.

Defn: Let X be a r.v. with $E[X] = \mu$. The variance of X is $Var(X) = E[(X-\mu)^2]$, often denoted σ^2.

Defn: The standard deviation of X is
$$\sigma = \sqrt{Var(X)}$$

Ex: $Var(Y) = E[(Y-\mu)^2]$, where $\mu = E[Y] = 0$
$$= E[Y^2] = \tfrac{1}{2}\cdot 1{,}000{,}000 + \tfrac{1}{2}\cdot 1{,}000{,}000 = 1{,}000{,}000$$
$$\sigma = \sqrt{Var(Y)} = 1000$$

Theorem: $\mathrm{Var}(X) = E[X^2] - (E[X])^2$.

Proof: Let $\mu = E[X]$

$$\mathrm{Var}(X) = E[(X-\mu)^2] = \sum_x (x-\mu)^2 p_X(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) p_X(x)$$

$$= \sum_x x^2 p_X(x) - 2\mu \sum_x x p_X(x) + \mu^2 \sum_x p_X(x)$$

$$= E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

Theorem: $\mathrm{Var}(aX+b) = a^2 \mathrm{Var}(X)$, for constants a, b

Proof: $\mathrm{Var}(aX+b) = E[((aX+b) - (a\mu+b))^2]$,

where $\mu = E[X]$

$$\mathrm{Var}(aX+b) = E[(a(X-\mu))^2]$$

$$= E[a^2(X-\mu)^2] = a^2 E[(X-\mu)^2]$$

$$= a^2 \mathrm{Var}(X).$$

In general, $\mathrm{Var}(X+Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y)$.

In particular, $\mathrm{Var}(X+X) = \mathrm{Var}(2X) = 4\mathrm{Var}(X) \neq \mathrm{Var}(X) + \mathrm{Var}(X)$

Defn: R.V.'s X and Y are independent iff

$$\forall x \forall y \; P(X=x \cap Y=y) = P(X=x)P(Y=y).$$

Ex: Flip a fair coin $2n$ times independently.

Let X be #heads in first n flips,

Y be #heads in last n flips, and

Z be #heads in all $2n$ flips.

X and Y are independent: exercise.

X and Z are dependent: $P(X=0) > 0$,

$P(Z=n+1) > 0$, but $P(X=0 \cap Z=n+1) = 0$ by

pigeonhole principle

<u>Theorem</u>: If X and Y are independent r.v.'s, then $E[XY] = E[X]E[Y]$.

<u>Proof</u>: $E[XY] = \sum_x \sum_y xy\, P(X=x \cap Y=y)$

$= \sum_x \sum_y xy\, P(X=x)P(Y=y)$ (ind.)

$= \sum_x x P(X=x) \left(\sum_y y\, P(Y=y) \right)$

$= \left(\sum_x x\, P(X=x) \right) \left(\sum_y y\, P(Y=y) \right)$

$= E[X]E[Y]$

<u>Theorem</u>: If X and Y are independent r.v.'s, then $Var(X+Y) = Var(X) + Var(Y)$.

<u>Proof</u>: $Var(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$

$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$

$= (E[X^2] + 2E[XY] + E[Y^2])$
$\quad - ((E[X])^2 + 2E[X]E[Y] + (E[Y])^2)$

$= (E[X^2] - (E[X])^2) + (E[Y^2] - (E[Y])^2)$
$\quad + (2E[X]E[Y] - 2E[X]E[Y])$ (ind)

$= Var(X) + Var(Y)$.