Defn: \[ E[g(X)] = \sum_x g(x) p_X(x) \]

**Linearity of Expectation**

**Theorem:** For any constants \( a \) and \( b \), \[ E[aX + b] = a E[X] + b. \]

**Proof:** \[ E[aX + b] = \sum_x (ax + b) p_X(x) \]
\[ = \sum_x (ax p_X(x) + b p_X(x)) \]
\[ = a \sum_x x p_X(x) + b \sum_x p_X(x) \]
\[ = a E[X] + b \]

**Ex:** A casino charges \$1 to play the following game: They flip a coin with \( P(\text{heads}) = \frac{1}{8} \) until it comes up heads. They pay you \$12 if it's heads. Do you expect to earn on this? Let \( X \) be the \# of flips until the first head. Your gain is \( 12X - 100 \)
\[ E[12X - 100] = 12 E[X] - 100 = 12 \cdot \frac{1}{8} - 100 = -4, \]
because \( X \) is a geometric random variable with prob \( \frac{1}{8} \) of heads.

**Theorem:** Let \( X \) and \( Y \) be two random variables, possibly dependent. Then \( E[X + Y] = E[X] + E[Y]. \)

**Proof:** Let \( X(s), Y(s) \) be values for outcome \( s \in \Omega. \)
\[ E[X + Y] = \sum_{s \in \Omega} (X(s) + Y(s)) p(s) \]
\[ = \sum_{s \in \Omega} X(s) p(s) + \sum_{s \in \Omega} Y(s) p(s) \]
\[ = E[X] + E[Y] \]
Ex: Let $X$ be the number of heads when a coin with $P(\text{heads}) = p$ is flipped $n$ times.
Let $X_i = 1$, if the $i$th flip is heads, for $1 \leq i \leq n$.

(An indicator r.v. has values 0 or 1.)

$$E[X_i] = 1 \cdot P(\text{heads}) + 0 \cdot P(\text{tails}) = p$$

$$X = \sum_{i=1}^n X_i$$

$$E[X] = E\left[ \sum_{i=1}^n X_i \right] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n p = np$$

Ex: Shuffle 4 aces, deal 2 of them into 2 piles.
Let $X$ be the number of spades in the 2 piles. $E[X]$?

Let $X_i = 1$, if the $i$th pile has 2 A, for $i \in \{1, 2\}$

Are $X_1 = 1$ and $X_2 = 1$ independent? No:

$$P(X_2 = 1|X_1 = 1) = 0 \neq P(X_2 = 1) = \frac{1}{4}$$

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(X_i = 1) = \frac{1}{4}$$

$$E[X] = E[X_1 + X_2] = E[X_1] + E[X_2] = 2 \cdot \frac{1}{4} = \frac{1}{2}$$