<u>Defn</u>: $E[g(X)] = \sum_x g(x)\, p_X(x)$

<u>Linearity of Expectation</u>

<u>Theorem</u>: For any constants a and b,

$$E[aX+b] = aE[X]+b.$$

<u>Proof</u>:

$$E[aX+b] = \sum_x (ax+b)\, p_X(x)$$

$$= \sum_x \left(ax\, p_X(x) + b\, p_X(x)\right)$$

$$= a\sum_x x\, p_X(x) + b\sum_x p_X(x)$$

$$= aE[X] + b$$

<u>Ex</u>: A casino charges \$1 to play the following game: They flip a coin with $P(\text{heads}) = 1/8$ until it comes up heads. They pay you 12¢ per flip. Do you expect to win or lose? Let X be the # of flips until the first head. Your gain is $12X - 100$

$$E[12X-100] = 12E[X] - 100 = 12 \cdot \frac{1}{1/8} - 100 = -4,$$

because X is a geometric random variable with prob $1/8$ of heads.

<u>Theorem</u>: Let X and Y be two random variables, <u>possibly dependent</u>. Then $E[X+Y] = E[X]+E[Y]$.

<u>Proof</u>: Let $X(s), Y(s)$ be values for outcome $s \in \Omega$.

$$E[X+Y] = \sum_{s\in\Omega} (X(s)+Y(s))P(s)$$

$$= \sum_{s\in\Omega} X(s)P(s) + \sum_{s\in\Omega} Y(s)P(s)$$

$$= E[X] + E[Y]$$

<u>Ex</u>: Let X be # of heads when a coin with $P(\text{heads}) = p$ is flipped n times.

Let $X_i = \begin{cases} 1, & \text{if } i^{\text{th}} \text{ flip is heads} \\ 0, & \text{otherwise} \end{cases}$, for $1 \le i \le n$.

(An indicator r.v. has values 0 or 1.)

$$E[X_i] = 1 \cdot P(\text{heads}) + 0 \cdot P(\text{tails}) = p$$

$$X = \sum_{i=1}^{n} X_i$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

<u>Ex</u>: Shuffle 4 aces, deal 2 of them into 2 piles.

Let X be the # of spades in the 2 piles. $E[X]$?

Let $X_i = \begin{cases} 1, & \text{if } i^{\text{th}} \text{ pile has } \spadesuit A \\ 0, & \text{otherwise} \end{cases}$, for $i \in \{1, 2\}$

Are $X_1 = 1$ and $X_2 = 1$ independent? No:

$$P(X_2 = 1 \mid X_1 = 1) = 0 \neq P(X_2 = 1) = 1/4$$

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(X_i = 1) = \frac{1}{4}$$

$$E[X] = E[X_1 + X_2] = E[X_1] + E[X_2] = 2 \cdot \frac{1}{4} = \frac{1}{2}$$