Random Variables

Defn: A random variable is a numeric function of the outcome.

Ex 1 # heads when 20 coins are flipped.

2 total of 2 die rolls

3. # of coin tosses until first head.

Suppose coin has prob p of heads. Let X be # of independent tosses ~~until~~ up to and including the first head.

$$P(X=1) = P(H) = p$$
$$P(X=2) = P(TH) \stackrel{I}{=} (1-p)p$$
$$P(X=i) = P(T^{i-1}H) = (1-p)^{i-1}p$$

Defn: If a random variable has a countable number of possible values, it is called discrete. (Countable means either finite, or in one-to-one correspondence with the nonnegative integers.)

Defn: If X is a discrete r.v. with values in a countable set T, the probability mass function (pmf) of X is

$$p_X(a) = \begin{cases} P(X=a), & \text{if } a \in T \\ 0, & \text{otherwise} \end{cases}$$

Note: $\sum_{a \in T} p_X(a) = 1$.

Defn. For a discrete r.v. X with pmf $p_X(a)$, the expectation (or expected value or mean) of X is $E[X] = \sum_a a\, p_X(a)$.

In the case of equally likely outcomes, $E[X]$ is just the average of the possible values of X. In general, $E[X]$ is a weighted average, where each value is weighted by its probability.

Let X be the number of independent flips up to and including the first heads of a coin with prob p of heads. X is called the ~~the geometric~~ geometric r.v.

PMF for X is $p_X(i) = (1-p)^{i-1} p$.

$$E[X] = \sum_{i=1}^{\infty} i\, p_X(i) = \sum_{i=1}^{\infty} i(1-p)^{i-1} p = p \sum_{i=1}^{\infty} i(1-p)^{i-1}$$

Calculus: $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} = (1-x)^{-1}$ if $|x|<1$.

Take derivative wrt. x:

$$\sum_{i=1}^{\infty} i x^{i-1} = \cancel{+}(1-x)^{-2} = \frac{1}{(1-x)^2}$$

Substitute $x = 1-p$:

$$E[X] = p \sum_{i=1}^{\infty} i(1-p)^{i-1} = p \cdot \frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

Ex $p = \frac{1}{2}$: $E[X] = 2$

$p = \frac{1}{10}$: $E[X] = 10$

Ex: $E[X] = \frac{5}{6}(+1) + \frac{1}{6}(-2) = \frac{1}{2}$

In 5 cases, I gain 1 game point.

In 1 case, I lose 2 game points.

X = # of game points I gain.

Choose the play with the greatest value of $E[X]$.

<u>Defn</u>: $E[g(X)] = \sum_x g(x)\, p_X(x)$