Defn: Two events $E$ and $F$ are independent iff $P(\text{E}\cap \text{F}) = P(\text{E})P(\text{F})$. Otherwise they are dependent.

Ex: Roll 2 fair dice, yielding values $D_1$ and $D_2$. Let $E$ be "$D_1 = 1$", $F$ be "$D_1 + D_2 = 7$", $G$ be "$D_1 + D_2 = 5$".

$P(E) = \frac{1}{6}$, $P(F) = \frac{6}{36} = \frac{1}{6}$

$P(\text{E}\cap \text{F}) = P(\text{D}_1 = 1 \cap \text{D}_2 = 6) = \frac{1}{36} = P(\text{E})P(\text{F})$,

so $E$ and $F$ are independent.

$P(G) = \frac{4}{36} = \frac{1}{9}$

$P(\text{E}\cap \text{G}) = P(\text{D}_1 = 1 \cap \text{D}_2 = 4) = \frac{1}{36} \neq P(\text{E})P(\text{G}) = \frac{1}{18}$

so $E$ and $G$ are dependent.

Defn: $E_1, E_2, \ldots, E_n$ are independent iff for every subset $S \subseteq \{1, 2, \ldots, n\}$

$P(\bigcap_{i \in S} E_i) = \prod_{i \in S} P(E_i)$.

Theorem: If $E$ and $F$ are events and $P(F) > 0$, then $E$ and $F$ are independent iff $P(E|F) = P(E)$.

Proof:

$\implies$: If $E$ and $F$ are independent, then

$P(E)P(F) = P(\text{E}\cap \text{F}) = P(\text{E}|\text{F})P(\text{F})$, so dividing by $P(F)$ gives $P(E) = P(E|F)$.

$\impliedby$: If $P(E|F) = P(E)$ then

$P(\text{E}\cap \text{F}) = P(E)P(F) = P(E)P(F)$,

so $E$ and $F$ are independent.
If \( E \cap F = \emptyset \) and \( P(E) \neq 0 \) and \( P(F) \neq 0 \), then \( E \) and \( F \) must be dependent, because
\[
P(E \cap F) = 0 \neq P(E)P(F)
\]

Suppose a (biased) coin comes up heads with probability \( p \). Suppose it is flipped \( n \) times independently.
\[
P(n \text{ heads}) = p^n
\]
\[
P(\text{first } k \text{ are heads and the rest tails}) = p^k (1-p)^{n-k}
\]
\[
P(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}
\]

\( n \) slots, \( k \) heads
Network failure

Suppose we have $n$ routers in parallel, where the $i$th router fails with prob. $p_i$ independently of the others.

$$P(A \text{ can communicate with } B) = 1 - P(\text{all } n \text{ fail}) = 1 - p_1 p_2 \cdots p_n.$$  

Suppose they are in series

$$A \quad \cdots \quad B$$

$$\quad P_1 \quad P_2 \quad P_n$$

$$P(A \text{ can communicate with } B) = P(\text{none fail}) = (1-p_1)(1-p_2) \cdots (1-p_n).$$

Gambler's Ruin

$A$ has $\$i$ and $B$ has $\$N-i$.

Flip a fair coin. If $H$, $A$ pays $B$ $\$1$.
If $T$, $B$ pays $A$ $\$1$. Whoever gets all $\$N$ wins.

$$0 \quad 1 \quad 2 \quad 3 \quad i-1 \quad i \quad i+1 \quad N-1 \quad N$$
The game is a random walk on that line. Let $E_i$ be the event that $A$ wins the game starting with $i$. \[ P(E_i) = ? \]