<u>Defn</u>: Two events E and F are <u>independent</u> iff $P(E\cap F) = P(E)P(F)$. Otherwise they are <u>dependent</u>.

<u>Ex</u>: Roll 2 fair dice, yielding values D_1 and D_2. Let E be "$D_1 = 1$", F be "$D_1 + D_2 = 7$", G be "$D_1 + D_2 = 5$"

$P(E) = 1/6$, $P(F) = 6/36 = 1/6$

$P(E\cap F) = P(D_1 = 1 \cap D_2 = 6) = 1/36 = P(E)P(F)$, so E and F are independent.

$P(G) = 4/36 = 1/9$

$P(E\cap G) = P(D_1 = 1 \cap D_2 = 4) = 1/36 \neq P(E)P(G) = 1/54$, so E and G are dependent.

<u>Defn</u>: $E_1, E_2, \ldots, E_n$ are <u>independent</u> iff for every subset $S \subseteq \{1, 2, \ldots, n\}$

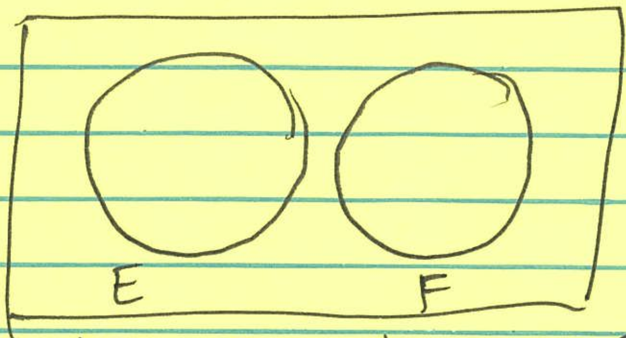
$$P\left(\bigcap_{i\in S} E_i\right) = \prod_{i\in S} P(E_i).$$

<u>Theorem</u>: If E and F are events and $P(F) > 0$, then E and F are independent iff $P(E|F) = P(E)$.

<u>Proof</u>:

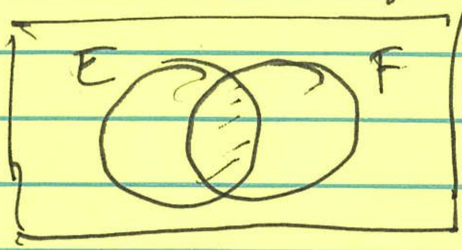
$\Rightarrow$: If E and F are indep, then $P(E)P(F) = P(E\cap F) = P(E|F)P(F)$, so dividing by $P(F)$ gives $P(E) = P(E|F)$.

$\Leftarrow$: If $P(E|F) = P(E)$ then $P(E\cap F) = P(E|F)P(F) = P(E)P(F)$, so E and F are independent.



Ω $\qquad E\cap F=\emptyset$

If $E\cap F=\emptyset$ and $E\neq\emptyset$ and $F\neq\emptyset$, then E and F must be dependent, because $P(E\cap F)=0\neq P(E)P(F)$



Suppose a (biased) coin comes up heads with probability p. Suppose it is flipped n times independently.

$P(n \text{ heads}) = p^n$

$P(\text{first } k \text{ are heads and the rest tails}) = p^k(1-p)^{n-k}$

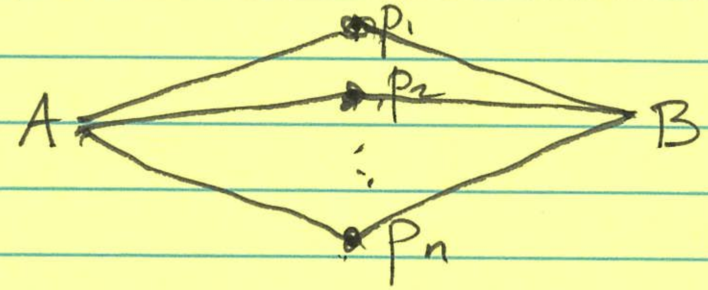
$P(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$

_ H H _ _ H _ _ H

n slots, k heads

Network failure

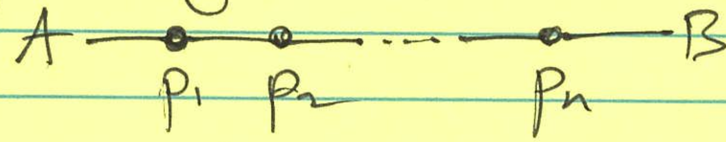
Suppose we have n routers in parallel, where the i^{th} router fails with prob. p_i independently of the others.



P(A can communicate with B)

$= 1 - P(\text{all } n \text{ fail}) = 1 - p_1 p_2 \cdots p_n$.

Suppose they are in series



P(A can communicate with B) $= P$(none fail)

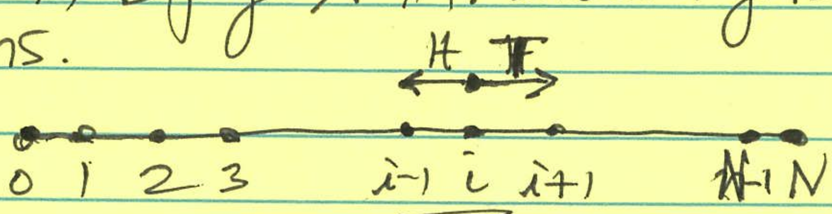
$= (1-p_1)(1-p_2)\cdots(1-p_n)$.

Gambler's Ruin

A has \$$i$ and B has \$$(N-i)$.

Flip a fair coin. If H, A pays B \$1.

If T, B pays A \$1. Whoever gets all \$N wins.



The game is a random walk on that line. Let E_i be the event that A wins the game starting with \$$i$. $P(E_i) = ?$