$$P(E|F) = \frac{P(E\cap F)}{P(F)}$$

Chain Rule: $P(E\cap F) = P(E|F)P(F)$

Generalization: $P(E_1 \cap E_2 \cap \cdots \cap E_n) =$

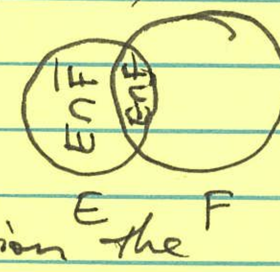
$P(E_1)P(E_2|E_1)P(E_3|E_1\cap E_2)\cdots P(E_n|E_1\cap E_2 \cap \cdots \cap E_{n-1})$

<u>Law of Total Probability</u>: If E and F are events, then $P(E) = P(E|F)P(F) + P(E|\overline{F})P(\overline{F})$

<u>Proof</u>: $P(E) = P((E\cap F)\cup(E\cap\overline{F}))$

$= P(E\cap F) + P(E\cap\overline{F})$ (Ax.3)

$= P(E|F)P(F) + P(E|\overline{F})P(\overline{F})$



<u>Generalization</u>: If $F_1, F_2, \ldots, F_n$ partition the sample space Ω, then

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i).$$

<u>Ex</u>: Sally will take either Phys or Chem. She will get an A in Phys with prob. 3/4 and an A in Chem with prob. 3/5. She flips a fair coin to decide which to take. Let A be the event that she gets an A.

$P(A) = P(A|\text{Phys})P(\text{Phys}) + P(A|\text{Chem})P(\text{Chem})$

$= \frac{3}{4}\cdot\frac{1}{2} + \frac{3}{5}\cdot\frac{1}{2} = \frac{27}{40}.$

<u>Bayes' Theorem</u> (Rev. Thomas Bayes, c. 1701–1761):

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E)} \text{ for events } E \text{ and } F.$$

<u>Proof</u>: $P(F|E) = \dfrac{P(E \cap F)}{P(E)} = \dfrac{P(E|F)\,P(F)}{P(E)}$

<u>Corollary</u>: $P(F|E) = \dfrac{P(E|F)\,P(F)}{P(E|F)\,P(F) + P(E|\bar{F})\,P(\bar{F})}$

Why it's useful: Reverses the conditioning.

<u>Ex</u>: 60% of email is spam.
90% of spam has a forged header.
20% of nonspam has a forged header.
Let F = forged header, J = spam.
What is $P(J|F)$? We're given $P(F|J) = 0.9$.

$$P(J|F) = \frac{P(F|J)\,P(J)}{P(F|J)\,P(J) + P(F|\bar{J})\cdot P(\bar{J})}$$

$$= \frac{0.9 \times 0.6}{0.9 \times 0.6 + 0.2 \times (1 - 0.6)} \approx 0.871$$

<u>prior probability</u> or <u>prior</u> is 0.6
<u>posterior</u> is 0.871

<u>Ex</u>: Paternity testing

Child has (A, a) gene pair : event $B_{A,a}$.

Mother has (A, A).

Two possible fathers: $F_1 = (a,a)$, $F_2 = (A,a)$

$P(F_1) = p$, $P(F_2) = 1-p$.

$$P(F_1 \mid B_{A,a}) = \frac{P(B_{A,a} \mid F_1) P(F_1)}{P(B_{A,a} \mid F_1) P(F_1) + P(B_{A,a} \mid F_2) P(F_2)}$$

$$= \frac{1 \cdot p}{1 \cdot p + \frac{1}{2}(1-p)} = \frac{2p}{p+1} \geq \frac{2p}{1+1} = p$$

E.g., if prior $p = \frac{1}{2}$, the posterior is $\frac{2}{3}$.

<u>Defn</u>: Two events E and F are <u>independent</u> iff $P(E \cap F) = P(E)P(F)$.