\[ P(E | F) = \frac{P(E \cap F)}{P(F)} \]

Chain Rule: \( P(E \cap F) = P(E | F) P(F) \)

Generalization: \( P(E, E_2 \cap \cdots \cap E_n) = P(E) P(E_2 | E) P(E_3 | E, E_2) \cdots P(E_n | E, E_2 \cap \cdots \cap E_{n-1}) \)

Law of Total Probability: If \( E \) and \( F \) are events, then \( P(E) = P(E | F) P(F) + P(E | F^c) P(F^c) \)

Proof: \( P(E) = P((E \cap F) \cup (E \cap F^c)) \)
\[ = P(E \cap F) + P(E \cap F^c) \quad (\text{Ax. } 3) \]
\[ = P(E | F) P(F) + P(E | F^c) P(F^c) \]

Generalization: If \( F_1, F_2, \ldots, F_n \) partition the sample space \( \Omega \), then
\[ P(E) = \sum_{i=1}^{n} P(E | F_i) P(F_i) \]

Ex: Sally will take either Phys or Chem. She will get an A in Phys with prob. \( \frac{3}{4} \) and an A in Chem with prob. \( \frac{3}{5} \). She flips a fair coin to decide which to take. Let \( A \) be the event that she gets an A.

\[ P(A) = P(A | \text{Phys}) P(\text{Phys}) + P(A | \text{Chem}) P(\text{Chem}) \]
\[ = \frac{3}{9} \cdot \frac{10}{2} + \frac{3}{5} \cdot \frac{2}{2} = \frac{27}{40} \]
Bayes' Theorem (Rev. Thomas Bayes, c. 1701–1761):

\[ P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E)} \quad \text{for events } E \text{ and } F. \]

Proof:

\[ P(F \mid E) = \frac{P(E \cap F)}{P(E)} = \frac{P(E \mid F)P(F)}{P(E)} \]

Corollary:

\[ P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid F')P(F')} \]

Why it's useful: Reverses the conditioning.

Example:

60% of email is spam.

90% of spam has a forged header.

20% of non-spam has a forged header.

Let \( F \) = forged header, \( J \) = spam.

What is \( P(J \mid F) \)? We're given \( P(F \mid J) = 0.9 \).

\[ P(J \mid F) = \frac{P(F \mid J)P(J)}{P(F \mid J)P(J) + P(F \mid J')P(J')} \]

\[ = \frac{0.9 \times 0.6}{0.9 \times 0.6 + 0.2 \times (1-0.6)} \approx 0.871 \]

Prior probability or prior is 0.6

Posterior is 0.871
Ex: Paternity testing

Child has \((A,a)\) gene pair: event \(B_{A,a}\).
Mother has \((A,A)\).

Two possible fathers: \(F_1 = (a,a), F_2 = (A,a)\).

\[ P(F_1) = p, \quad P(F_2) = 1 - p. \]

\[ P(F_1 | B_{A,a}) = \frac{P(B_{A,a} | F_1) P(F_1)}{P(B_{A,a} | F_1) P(F_1) + P(B_{A,a} | F_2) P(F_2)}. \]

\[ = \frac{1 \cdot p}{1 - p + \frac{1}{2} (1 - p)} = \frac{2p}{p + 1} \geq \frac{2}{3} = p. \]

E.g., if prior \(p = \frac{1}{2}\), the posterior is \(\frac{2}{3}\).

Defn: Two events \(E\) and \(F\) are independent if \(P(\text{E} \cap \text{F}) = P(\text{E}) P(\text{F})\).