Ex: Assume 365 birthdays are equally likely.
What is the probability that of $n \leq 365$ people, none share the same birthday?

$|\Omega| = 365^n$, $|E| = P(365, n) = 365 \cdot 364 \cdots (365-n+1)$

$$p_n = P(\text{no shared birthday among the } n \text{ people}) = \frac{365 \cdot 364 \cdots (365-n+1)}{365^n}$$

Some values:

$n = 23$: $p_{23} < 0.5$

$n = 100$: $p_{100} < \frac{1}{3 \times 10^{15}}$

Ex: n chips manufactured, d of them are defective. k chips are selected randomly for testing, where $k \leq n-d$. What is

$P(k$ selected chips contain <u>some</u> defective chips$)$?

Use complementing:
Let E be event that none of the k chips are defective.

$$P(E) = \frac{\binom{n-d}{k}}{\binom{n}{k}}$$

$$P(\geq 1 \text{ defective chips}) = 1 - \frac{\binom{n-d}{k}}{\binom{n}{k}}$$

Let E and F be events in a sample space Ω.
Conditional probability of E given F, written $P(E|F)$, where $F \neq \emptyset$, is the probability that E occurs, given that F was observed. Sample space is reduced to F, and event is reduced to $E \cap F$.
With equally likely outcomes,

$$P(E|F) = \frac{|E \cap F|}{|F|}$$

$$= \frac{|E \cap F| / |\Omega|}{|F| / |\Omega|} = \frac{P(E \cap F)}{P(F)}$$

Ex. Roll a fair die. What is $P(5|\text{odd})$?

$E = \{5\}$, $F = \{1,3,5\}$.

1. From counting, $P(5|\text{odd}) = \dfrac{|\{5\} \cap \{1,3,5\}|}{|\{1,3,5\}|} = \dfrac{1}{3}$
2. From probabilities,

$$P(5|\text{odd}) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{1/6}{3/6} = \frac{1}{3}$$

Ex: Assume ♤J is face-up. Let Ω be all the ways of dealing a 5-card hand to each of you and your opponent. Let

Y = you are dealt 0 trumps,
O = opponent is dealt 0 trumps.

$$P(O|Y) = \frac{|O \cap Y|}{|Y|} = \frac{\binom{15}{5}\binom{10}{5}}{\binom{15}{5}\binom{14}{5}}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 / 5!}{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 / 5!} \approx 0.126$$

Compare to $P(O) \approx 0.258$

$P(\text{Opp is dealt} \geq 1 \text{ trump} | Y) = 1 - P(O|Y) \approx 0.874$

Outcomes not all equally likely.

Ex: Let $\Omega = \{0, 1, 2\}$, the number of heads when two fair coins are flipped.

$P(0) = P(2) = 1/4, \; P(1) = 1/2$

General definition of conditional probability

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Ex: Let E be the event "2 heads" and F be the event "≥ 1 head"

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$$