Sample space Ω, event $E \subseteq \Omega$

Ex: ≥ 1 H in 2 coinflips: $E = \{HH, HT, TH\}$
odd roll of a die: $E = \{1, 3, 5\}$
5-card Schnapsen hand with no trump: $|E| = \binom{15}{5}$

Defn: E and F are mutually exclusive iff $E \cap F = \emptyset$.
(or disjoint).

Axioms of Probability: There is a function P that assigns a real number $P(E)$ to every event E, satisfying:

(1) $P(E) \geq 0$,
(2) $P(\Omega) = 1$, and
(3) If E and F are mutually exclusive, then
$$P(E \cup F) = P(E) + P(F).$$

Implications of the axioms:

(a) $P(\overline{E}) = P(\Omega - E) = 1 - P(E)$, because
$$1 \underset{\text{Ax}(2)}{=} P(\Omega) = P(E \cup \overline{E}) \underset{\text{Ax}(3)}{=} P(E) + P(\overline{E})$$

(b) If $E \subseteq F$, then $P(E) \leq P(F)$, because
$$P(F) = P(E \cup (F - E)) \underset{\text{Ax}(3)}{=} P(E) + P(F - E) \underset{\text{Ax}(1)}{\geq} P(E) + 0 = P(E)$$

(c) $P(E) \leq 1$, because $E \subseteq \Omega$ and $P(\Omega) = 1$ & implic. (b).

(d) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. (several applications of ax (3))

Special case: equally likely outcomes: Ω is finite, and if $a \in \Omega$, then $P(a) = 1/|\Omega|$.

Ex: One flip of a fair coin: $P(H) = P(T) = 1/2$

Two flips of a fair coin: $P(HH) = P(HT) = P(TH) = P(TT) = 1/4$

Roll of ~~of~~ a fair die: $P(1) = \cdots = P(6) = 1/6$.

Let $E \subseteq \Omega$. Then $P(E) = P\left(\bigcup_{a\in E} \{a\}\right) = \sum_{a\in E} P(a)$ (Ax. 3)

$$= \sum_{a\in E} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}$$

Ex: If E is ≥ 1 head in 2 flips of a fair coin, then

$$P(E) = \frac{|E|}{|\Omega|} = \frac{3}{4}$$

Ex: Assume ♠J is face-up. Assume any 5-card hand is equally likely.

$$P(\text{no trump in the initial hand}) = \frac{\binom{15}{5}}{\binom{19}{5}} = \frac{3003}{11{,}628} \approx 0.258$$

$$P(\geq 1 \text{ trump in the initial hand}) \approx 1 - 0.258 = 0.742$$

Ex: Assume your 5-card hand is dealt before a trump is turned up

$$P(\geq 1 \text{ marriage in initial hand}) = \frac{\binom{4}{1}\binom{18}{3} - \binom{4}{2}\binom{16}{1}}{\binom{20}{5}} = \frac{|E|}{|\Omega|}$$

$$\approx 0.204$$

$$\underbrace{\binom{4}{1}}_{\substack{\#\text{ of ways to}\\ \text{pick marriage suit}}}\underbrace{\binom{18}{3}}_{\substack{\#\text{ of ways to}\\ \text{deal 3 more cards}}}$$

E = ~~##~~ hands with ≥ 1 marriage

Ω = hands of any kind

Ex: Assume 365 birthdays are equally likely. What is the probability that of $n \leq 365$ people, none share the same birthday?

$|\Omega| = 365^n$, $|E| = P(365, n)$