Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, for any nonnegative integer n and any x, y.

Proof: $(x+y)^n = \underbrace{(x+y)(x+y)(x+y) \cdots (x+y)}_{n \text{ factors}}$

From each factor, choose either x or y to multiply in, and then collect all the terms $x^k y^{n-k}$. How many are there?

$$\underbrace{\underline{x}\ \_\ \underline{x}\ \_\ \_\ \underline{x}\ \underline{x}\ \_\ \_}_{n \text{ factors, } k \text{ of them are } x}$$

How many ways? $\binom{n}{k}$. So the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$.

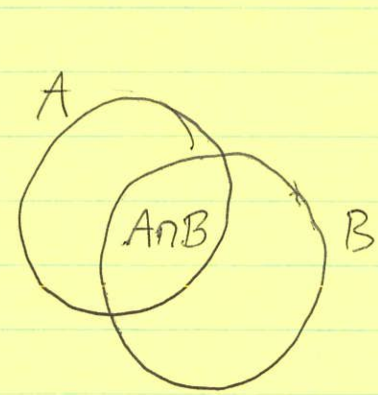
Corollary: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof: $2^n = (1+1)^n \underset{\substack{\uparrow \\ \text{Binomial Theorem}}}{=} \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

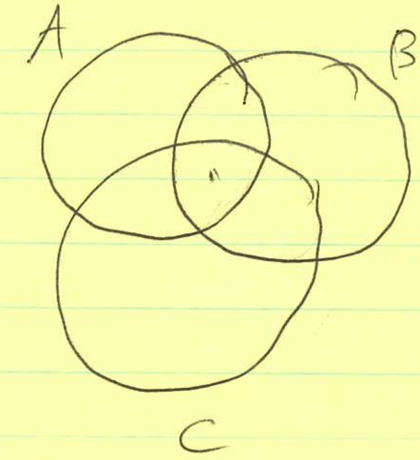
Both sides of the corollary count the number of subsets of a set of size n.

Inclusion-Exclusion Principle:

$|A \cup B| = |A| + |B| - |A \cap B|$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

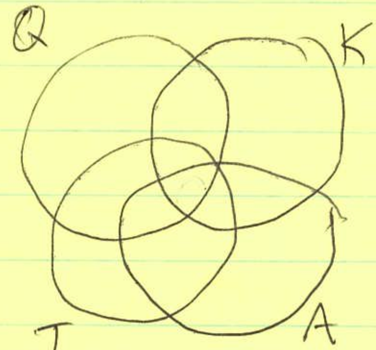


In general: + singles − pairs + triples − quads + ...

Recall our Schnapsen example: ♢J face-up. How many initial 5-card hands have ≥ 1 trump?

$$\binom{4}{1}\binom{18}{4} - \binom{4}{2}\binom{17}{3} + \binom{4}{3}\binom{16}{2} - \binom{4}{4}\binom{15}{1}$$

$$12{,}240 - 4080 + 480 - 15 = 8625$$



<u>Pigeonhole Principle</u>: If you have $n+1$ pigeons in n pigeonholes, then some pigeonhole must have >1 pigeon.

<u>Ex</u>:

5×5 chessboard.
1 flea on each square. When you ring a bell, each flea jumps to a random adjacent square.
Some square must have > 1 flea.

13 black squares. Pigeons = 13 fleas on black.
Pigeonholes = 12 white squares.

Intro to probability

1. **Sample space**: set Ω of possible "outcomes" of an experiment.

   Ex: Coin flip: $\Omega = \{H, T\}$

   2 coin flips: $\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$.

   Die roll: $\Omega = \{1,2,3,4,5,6\}$.

   5-card Schnapsen hands, assuming ♠J is face-up:

   $|\Omega| = \binom{19}{5}$.

2. **Event**: any $E \subseteq \Omega$

   Ex: ≥ 1 head in 2 coin flips: $E = \{(H,H), (H,T), (T,H)\}$