CSE 312: Foundations of Computing II
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Lecture Topics: 3.4 Zoo of Discrete RVs I

[**Tags:** PSet2 Q9, Binomial RV, Linearity of Expectation]

1. To determine whether they have llama flu, 1000 people have their blood tested. However, rather than testing each individual separately (1000 tests is quite costly), it is decided to use a group testing procedure:
    - **Phase 1**: First, place people into groups of 5. The blood samples of the 5 people in each group will be pooled and analyzed together. If the test is positive (at least one person in the pool has llama flu), continue to Phase 2. Otherwise send the group home. 200 of these pooled tests are performed.
    - **Phase 2**: Individually test each of the 5 people in the group. 5 of these individual tests are performed per group in Phase 2.

    Suppose that the probability that a person has llama flu is 5% for all people, independently of others, and that the test has a 100% true positive rate and 0% false positive rate (note that this is unrealistic). Using this strategy, compute the expected total number of blood tests (individual and pooled) that we will have to do across Phases 1 and 2.

Solution: Watch lecture ☺

[**Tags:** Binomial RV, Geometric RV, Negative Binomial RV]

2. Suppose Sammy the Beginner Tennis Player must practice his one-handed backhand in tennis. His goal is to hit them like Roger Federer does, so he does 1000 practice swings every day.
   a. Sammy misses the ball every time he swings with probability 0.8, independently of other swings. If he does manage to hit the ball with his swing, the probability it actually goes over the net is 0.1. What is the probability that when a ball comes, he hits it over the net?
   b. A day is a "huge success" if he hits it over the net at least fifty times that day. What is the probability of a huge success?
   c. Let p be your answer from part (b), the probability a single day is a huge success. Let X be the number of days he takes up to and including his first huge success. What is the PMF of X, $p_X(k)$?
   d. Let Y be the number of days up to and including his ninth huge success. What is the PMF of Y, $p_Y(k)$?
   e. What is $E[Y]$? (Hint: $E[X] = \frac{1}{p}$ from part (c). Try using linearity of expectation!)

Solution:

a. Let M be the event he misses, and N be the event it goes over the net. Then,
$$P(N) = P(M^C \cap N) = P(M^C)P(N|M^C) = 0.2 \cdot 0.1 = 0.02$$
b. Let X be the number of times he hits it over the net in a day. Then, $X \sim Bin(n = 1000, p = 0.02)$, so
$$P(X \geq 50) = \sum_{k=50}^{1000} \binom{1000}{k} 0.02^k (1 - 0.02)^{1000-k}$$
c. The first $k - 1$ trials must be failures and the last trial is a huge success. So
$$p_X(k) = (1 - p)^{k-1} p$$
d. In the first $k - 1$ trials, he must get 8 huge successes and $k - 9$ failures (these can happen in any order). Then he must finish with a huge success. Hence,
$$p_Y(k) = \binom{k-1}{9-1} p^8 (1-p)^{k-9} \cdot p = \binom{k-1}{8} p^9 (1-p)^{k-9}$$
e. Then $Y = X_1 + \cdots + X_9$ where X_i is the number of trials up to and including the i^{th} huge success from the $(i-1)^{st}$ huge success. Each X_i has the same distribution as in part (c) with $E[X_i] = \frac{1}{p}$, so $E[Y] = \sum_{i=1}^{9} E[X_i] = 9 \cdot \frac{1}{p} = \frac{9}{p}$.

[**Tags:** Bernoulli RV, Linearity of Expectation]

3. Suppose there is a network of n prisoners, and each pair of prisoners are enemies with probability p, independently of other pairs. You know the saying "The enemy of my enemy is my friend" (two prisoners are friends if there exists at least one other prisoner who is an enemy to both and they can be friends even if they are enemies). What is the expected number of friendships between pairs of the n prisoners?

Solution: Let $X_i = 1$ if the i^{th} pair are friends, for $i = 1, \dots, m$ where $m = \binom{n}{2}$. Then, the expected number of friendships is $X = \sum_{i=1}^{m} X_i$.

For a particular pair to be friends, they need to have a common enemy from the other $n-2$ prisoners.

$$P(\geq 1\ common\ enemy) = 1 - P(no\ common\ enemies)$$
$$= 1 - P(both\ not\ enemies\ with\ n - 2\ people)$$
$$= 1 - P(both\ not\ enemy\ with\ one\ person)^{n-2}$$
$$= 1 - \left(1 - P(both\ enemies\ with\ one\ person)\right)^{n-2}$$
$$= 1 - (1 - p^2)^{n-2}$$

This could also be found by looking at the random variable Y_{12} to be the indicator of whether person 1 and person 2 are friends. This occurs if person 1 and person 2 have at least one common enemy. This implies that $Y_{12} \sim Bin(n-2, p^2)$ and then, $P(Y_{12} \geq 1) = 1 - P(Y_{12} = 0) = 1 - (1-p^2)^{n-2}$

Hence,

$$E[X_i] = P(X_i = 1) = 1 - (1 - p^2)^{n-2}$$

And

$$E[X] = \sum_{i=1}^{m} E[X_i] = \binom{n}{2}\left(1 - (1 - p^2)^{n-2}\right)$$