Lecture Topics: 2.3 Independence, 3.1 Discrete Random Variables Basics

[Tags: PSet1 Q10ac, Conditional Independence]

1. A website wants to detect if a visitor is a robot or a human. They give the visitor seven CAPTCHA tests that are hard for robots but easy for humans. If the visitor fails any of the tests, they are flagged as a robot. The probability that a human succeeds at a single test is 0.95, while a robot only succeeds with probability 0.3. Assume all tests are independent. The percentage of visitors on this website that are robots is 10%; all other visitors are human.
   a. If a visitor is actually a robot, what is the probability they get flagged (the probability they fail at least one test)?
   b. Compute the probability that a random visitor is flagged. (Helps with part (c)).

Solution: Watch lecture 😊.

[Tags: Independence, Random Variables, PMFs, Expectation, PSet2 Q8 (Similar)]

2. There are 3 people in Alex’s family; his mom, dad, and sister. Each family member decides whether or not they want to come to lunch in his social-distancing home restaurant, independently of the others.
   • Mom wants to come with probability 0.8.
   • Dad wants to come with probability 0.6.
   • Sister wants to come with probability 0.1.

Unfortunately, if all 3 of them want to come, he must turn one of them away 😞 since the restaurant capacity is 2 guests. Otherwise, he will take everyone that comes. Let $X$ be the number of customers that Alex serves at lunch.
   a. What is the range $\Omega_X$, the PMF $p_X(k)$, and the expectation $E[X]$?
   b. If he charges everyone who comes $10, but it costs him $50 to make all the food, what is his expected profit?

Solution:

a. The range is $\Omega_X = \{0, 1, 2\}$ since we can have anywhere from 0 to 2 people. By independence,

$$P(X = 0) = P(M^c, D^c, S^c) = P(M^c)P(D^c)P(S^c) = 0.2 \cdot 0.4 \cdot 0.9 = 0.072$$

$$P(X = 1) = P(M, D^c, S^c) + P(M^c, D, S^c) + P(M^c, D^c, S)$$
$$= 0.8 \cdot 0.4 \cdot 0.9 + 0.2 \cdot 0.6 \cdot 0.9 + 0.2 \cdot 0.4 \cdot 0.1 = 0.404$$

$$P(X = 2) = 1 - P(X = 0) - P(X = 1) = 0.524$$

$$p_X(k) = \begin{cases} 0.072, & k = 0 \\ 0.404, & k = 1 \\ 0.524, & k = 2 \end{cases}$$
\[ E[X] = \sum_{k \in \Omega_X} k \cdot p_X(k) = 0 \cdot 0.072 + 1 \cdot 0.404 + 2 \cdot 0.524 = 1.452 \]


[Tags: Chain Rule, Inclusion-Exclusion]

3. Suppose \( n \) people sit around a table. Each person orders a different dish, but the waiter did not mark positions unfortunately. He has the correct \( n \) dishes, but gives a random dish to each person (each of the \( n! \) assignments is equally likely). What is the probability that no one has the dish they ordered placed in front of them?

**Solution:** Go to recitation tomorrow!