CSE 312: Foundations of Computing II
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Lecture Topics: 1.2 More Counting, 1.3 No More Counting Please

[**Tags**: CC1.2 Q5, Stars and Bars]

1. A hypothetical Meme Awards Committee is picking the "best living meme" of 2019. How many ways can they give 10 indistinguishable (wink wink) nominations to Keanu Reeves, Will Smith and Ninja, assuming they each get at least one nomination? Hint: It's a variation of Stars & Bars in that it "forces a star between each bar". Because of the condition that "they each get at least one nomination", consider giving each one nomination first before distributing the remaining 7 nominations.

Solution: We give each of the 3 people one vote. Then, we assign the other 7 votes any way we like to the three people – hence we have 7 "stars" and 2 "bars". By the stars/bars method, there are $\binom{9}{2} = 36$ ways to distribute the votes.

[**Tags**: Sum Rule, Product Rule, Binomial Coefficients)

1. Suppose you have five quarters left and you want to take exactly two classes per quarter. You want to take CSE1, CSE2,..., CSE10, but CSE1 and CSE2 must both be taken before CSE3, which must be taken before CSE4. CSE1 and CSE2 can be taken in any order, or together. The other classes can be taken any quarter, in any order, and have no prerequisites. How many different schedules can be formed (assume the two classes in a quarter are unordered)?

Solution: There are two cases; either CSE1 and CSE2 are taken in the same quarter, or not.

Case 1: Same Quarter

Then, we have the sequence CSE1/2 → CSE3 → CSE4. There are $\binom{5}{3}$ ways to assign these to three of the five quarters, and only one valid ordering afterward (so multiply by 1). There are 6 courses CSE3 can be paired with, and 5 that CSE4 can be paired with. Finally, there are 4 classes remaining across two quarters. We have $\binom{4}{2}$ for assigning the first quarter, and no choices to make for the last (only one choice). So we have

$$\binom{5}{3} \cdot 6 \cdot 5 \cdot \binom{4}{2}$$

Case 2: Different Quarters

Then, we have the sequence CSE1→ CSE2→CSE3→CSE4 OR CSE2→CSE1→CSE3→CSE4. There are two choices (as listed above) for which ordering of CSE1/CSE2, and then $\binom{5}{4}$ ways to choose which 4 quarters host this sequence. There are 6 choices for which class is with CSE1, 5 for CSE2, 4 for CSE3, and 3 for CSE4. Finally, the last remaining quarter only has two classes left, so there are no choices (only one). So we have

$$2\binom{5}{4} \cdot 6 \cdot 5 \cdot 4 \cdot 3$$

We add these to get the result.

[**Tags**: PSet1 Q7, Combinatorial Proofs]

2. **Give combinatorial proofs of the following identities:**
   a. Prove that $\binom{n}{2} = \sum_{k=1}^{n-1} k$.
   b. Prove that $2^n - 1 = \sum_{i=0}^{n-1} 2^i$. (**Hint**: Imagine a tournament bracket)

Solution: Watch lecture ☺

[**Tags**: Complementary Counting, Inclusion-Exclusion, Multinomial Coefficients]

3. Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other (there are two E's, two N's, and two I's). For example, "INGREEDINT" is invalid because the two E's are adjacent. Repeat the question for the letters "AAAAABBB".

Solution: The overall set of ways to arrange INGREDIENT without restrictions is $|U| = \frac{10!}{2!2!2!}$, since we have three duplicates.

Let S be the set of those ways where no two identical letters are adjacent (this is what we want). Then S^C is the set of those ways where at least two identical letters are adjacent (meaning at least one of EE exists, NN exists, II exists). We shall find S by complementary counting: $|S| = |U| - |S^C|$, and we already have $|U|$ above.

If we let

T_E: the arrangements where the two E's are adjacent

T_N: the arrangements where the two N's are adjacent

T_I: the arrangements where the two I's are adjacent

Then, $S^C = T_E \cup T_N \cup T_I$. By inclusion-exclusion, (singles – doubles + triples – quads …)

$$|S^C| = |T_E| + |T_N| + |T_I| - |T_E \cap T_N| - |T_E \cap T_I| - |T_N \cap T_I| + |T_E \cap T_N \cap T_I|$$

Note that $|T_E| = |T_N| = |T_I|$ since there are two of each (so it's the same logic). Let's compute $|T_E|$: we must have both E's together, so let's treat them as one entity. Then, there are 9 entities total, so we get

$$\frac{9!}{2!\,2!}$$

Same for the doubles, they are all the same quantity. Let's compute $|T_E \cap T_N|$: we must have both E's and both N's together, so let's treat them as one entity. Then, there are 8 entities total, so we get

$$\frac{8!}{2!}$$

Finally, $|T_E \cap T_N \cap T_I|$ has 7 (distinct) entities, with $7!$ arrangements. Hence, our answer is

$$|S| = |U| - |S^C| = \frac{10!}{2!\,2!\,2!} - \left(3 \cdot \frac{9!}{2!\,2!} - 3 \cdot \frac{8!}{2!} + 7!\right)$$

For more examples and solutions, see https://courses.cs.washington.edu/courses/cse312/18sp/.