Lecture Topics: 7.3 Method of Moments Estimation, 7.4 Beta/Dirichlet Distributions

Tags: Estimation

1. Suppose $\mathbf{x} = (x_1, ..., x_n)$ are iid samples from the following distributions. Estimate the parameter(s) using your favorite technique (MLE or MoM). Hint: Use MoM.
   a. The $\text{Gamma}(r, \lambda)$ distribution. Estimate both $r$ and $\lambda$.
   b. The $\text{Rayleigh}(\sigma)$ distribution with density $f_X(x; \sigma) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, x \geq 0$ with expectation $\sigma \sqrt{\frac{\pi}{2}}$.

Solution:

a. We have $k = 2$ parameters to estimate. Let $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $\bar{x}^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2$. We set

$$E[X] = \bar{x}$$
$$E[X^2] = \bar{x}^2$$

For a $\text{Gamma}(r, \lambda)$ rv,

$$E[X] = \frac{r}{\lambda}$$
$$E[X^2] = Var(X) + E[X]^2 = \frac{r}{\lambda^2} + \left(\frac{r}{\lambda}\right)^2 = \frac{r(r+1)}{\lambda^2}$$

So we must solve the two equations:

$$\frac{r}{\lambda} = E[X] = \bar{x}, \quad \frac{r(r+1)}{\lambda^2} = E[X^2] = \bar{x}^2$$

Let’s divide the second equation by the first (note $\bar{x}^2 \neq \bar{x}^2$), then subtract $\frac{r}{\lambda} = \bar{x}$:

$$\frac{\bar{x}^2}{\bar{x}} = \frac{E[X^2]}{E[X]} = \frac{r(r+1)}{\lambda^2} \frac{\bar{x}}{r} = \frac{r+1}{\lambda} \rightarrow \frac{1}{\lambda} = \frac{r+1}{\lambda} - \frac{r}{\lambda} = \frac{\bar{x}^2}{\bar{x}} - \bar{x} = \frac{\bar{x}^2 - \bar{x}^2}{\bar{x}}$$

Hence, taking the reciprocal gives

$$\lambda = \frac{\bar{x}}{\bar{x}^2 - \bar{x}^2}$$

Then, since $\frac{r}{\lambda} = \bar{x}$ or equivalently $r = \lambda \bar{x}$, we get

$$\hat{r} = \hat{\lambda} \bar{x} = \frac{\bar{x}^2}{\bar{x}^2 - \bar{x}^2}$$

b. We set

$$\sigma \sqrt{\frac{2}{\pi}} = E[X] = \bar{x} \quad \rightarrow \quad \hat{\sigma} = \frac{\bar{x}}{\sqrt{\frac{2}{\pi}}}$$
Suppose we roll a (possibly unfair) 4-sided die 29 times. Then, the number of times each digit appears is \( X = (X_1, X_2, X_3, X_4) \sim \text{Mult}_4(n = 29, p) \), where \( p = (p_1, p_2, p_3, p_4) \) is unknown.

We happened to observe 5 ones, 7 twos, 6 threes, and 11 fours.

a. A \( \text{Beta}(\alpha_1, \beta_1) \) rv would be suitable to model our belief on \( p_1 \) (the probability of rolling a one) with what parameters \( \alpha_1, \beta_1 \)?

b. A \( \text{Beta}(\alpha_2, \beta_2) \) rv would be suitable to model our belief on \( p_2 \) (the probability of rolling a two) with what parameters \( \alpha_2, \beta_2 \)?

c. Let’s instead say we wanted to jointly model all the unknown parameters \( p \). A \( \text{Dirichlet}(\gamma) \) would be suitable, more efficient than modelling all four separately, and also enforce that \( \sum_{i=1}^4 p_i = 1 \). Which parameter vector \( \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \) would best model our belief?

**Solution:**

a. We have \( \text{Beta}(\alpha_1 = 6, \beta_1 = 25) \) since we saw \( \alpha_1 - 1 = 5 \) ones and \( \beta_1 - 1 = 24 = 7 + 6 + 11 \) other values.

b. We have \( \text{Beta}(\alpha_2 = 8, \beta_2 = 23) \) since we saw \( \alpha_2 - 1 = 7 \) twos and \( \beta_2 - 1 = 22 = 5 + 6 + 11 \) other values.

c. We have \( \text{Dirichlet}(\gamma_1 = 6, \gamma_2 = 8, \gamma_3 = 7, \gamma_4 = 12) \) since we saw \( \gamma_1 - 1 = 5 \) ones, \( \gamma_2 - 1 = 7 \) twos, \( \gamma_3 - 1 = 6 \) threes, and \( \gamma_4 - 1 = 12 \) fours.