CSE 312: Foundations of Computing II
Instructor: Alex Tsun
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Lecture Topics: 5.5 Convolution

[**Tags:** Convolution]

1. We'll practice some discrete convolutions.
   a. Let $X \sim Ber(p)$ and $Y \sim Ber(q)$ be independent, and $Z = X + Y$. Find Ω_Z and $p_Z(z)$.
   b. Let $X \sim Bin(n, p)$ and $Y \sim NegBin(r, p)$, and $Z = X + Y$. Let $n = 3, r = 5$. Find Ω_Z and $p_Z(z)$.

Solution:

a. We know $\Omega_X = \Omega_Y = \{0,1\}$, so $\Omega_Z = \{0,1,2\}$. By the convolution formula,

$$p_Z(z) = \sum_{x \in \Omega_X} p_X(x) p_Y(z - x)$$

We handle them one at a time:

$$p_Z(0) = p_X(0)p_Y(0-0) + p_X(1)\boldsymbol{p_Y(0-1)} = (1-p)(1-q) + p \cdot 0 = (1-p)(1-q)$$

$$p_Z(1) = p_X(0)p_Y(1-0) + p_X(1)p_Y(1-1) = (1-p)q + p \cdot (1-q) = q + p - 2pq$$

$$p_Z(2) = p_X(0)\boldsymbol{p_Y(2-0)} + p_X(1)p_Y(2-1) = (1-p) \cdot 0 + p \cdot q = pq$$

Note the two highlighted quantities were 0 because they were out of range!

b. In our solution here, we will do for the general case of undefined n and r.
   We have $\Omega_X = \{0,1, \dots, n\}$ and $\Omega_Y = \{r, r+1, \dots,\}$, so $\Omega_Z = \{r, r+1, \dots\}$.

We know from the chart that:

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & k \in \Omega_X \\ 0, & \text{otherwise} \end{cases}$$

$$p_Y(y) = \begin{cases} \binom{y-1}{r-1} p^r (1-p)^{y-r}, & k \in \Omega_Y \\ 0, & \text{otherwise} \end{cases}$$

Let $z \in \Omega_Z$. The convolution formula says:

$$p_Z(z) = \sum_{x \in \Omega_X} p_X(x) p_Y(z - x)$$

Since we are summing over Ω_X, we don't need to worry about $p_X(x)$ being 0. We need to ensure that $p_Y(z - x) > 0$ so we need to enforce $z - x \in \Omega_Y$ → that $z - x \geq r$ or equivalently $x \leq z - r$.

We need to handle two cases because if $z \geq n + r$, then this condition is guaranteed (since $x \leq n$ (binomial) and $n \leq z - r$. However, in the case that $z < n + r$ this is not guaranteed and we have to be careful to make sure that $x \leq z - r$.

Case 1: If $r \leq z < n + r$, then

$$p_Z(z) = \sum_{x=0}^{z-r} p_X(x) p_Y(z-x) = \sum_{x=0}^{z-r} \binom{n}{x} p^x (1-p)^{n-x} \binom{z-x-1}{r-1} p^r (1-p)^{z-x-r}$$

Case 2: If $z \geq n + r$, then

$$p_Z(z) = \sum_{x=0}^{n} p_X(x) p_Y(z-x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \binom{z-x-1}{r-1} p^r (1-p)^{z-x-r}$$

So our final PMF is

$$p_Z(z) = \begin{cases} \displaystyle\sum_{x=0}^{z-r} \binom{n}{x} p^x (1-p)^{n-x} \binom{z-x-1}{r-1} p^r (1-p)^{z-x-r}, & r \leq z < n + r \\ \displaystyle\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \binom{z-x-1}{r-1} p^r (1-p)^{z-x-r}, & z \geq n + r \\ 0, & z < r \end{cases}$$

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2. Mitchell and Alex are competing together in a 2-mile relay race. The time Mitchell takes to finish his mile (in hours) is $X \sim Exp(\lambda = 2)$ and the time Alex takes to finish his mile (in hours) is continuous $Y \sim Unif(0,1)$. Alex starts immediately after Mitchell finishes his mile, and their performances are independent. What is the distribution of the total time taken to complete this race?

Solution:

a. Let $Z = X + Y$ be the total time taken. Then, $\Omega_Z = [0, \infty)$ since $\Omega_X = [0, \infty)$ and $\Omega_Y = [0,1]$.

We know from the chart that:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}, \qquad f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $z \in \Omega_Z$. The convolution formula says:

$$f_Z(z) = \int_{\Omega_Y} f_Y(y) f_X(z-y) dy = \int_0^1 f_Y(y) f_X(z-y) dy$$

Since we are integrating dy, we don't need to worry about $f_Y(y)$ being 0. We should check when $f_X(z-y) > 0$. This happens when $z - y \geq 0$ as $\Omega_X = [0, \infty)$. That is, we need $z \geq y$.

We actually need to split into two cases: when $z \in [0,1]$ and $z \in [1, \infty)$: this is because when $z \in [1, \infty)$, we are guaranteed that $z \geq y$ (since $y \in [0,1]$), and when $z \in [0,1]$ we are not.

Case 1: Let $z \in [0,1]$. Then, since we need $y \leq z$:

$$f_Z(z) = \int_0^z f_Y(y) f_X(z-y) dy = \int_0^z 1 \cdot \lambda e^{-\lambda(z-y)} dy = 1 - e^{-\lambda z}$$

Case 2: Let $z \in [1, \infty)$. Then, $y \leq z$ always, so:

$$f_Z(z) = \int_0^1 1 \cdot \lambda e^{-\lambda(z-y)} dy = \left(e^{\lambda} - 1\right) e^{-\lambda z}$$

Hence,

$$f_Z(z) = \begin{cases} 1 - e^{-\lambda z}, & z \in [0,1] \\ \left(e^{\lambda} - 1\right) e^{-\lambda z}, & z \in [1, \infty) \\ 0, & \text{otherwise} \end{cases}$$