Lecture Topics: 5.5 Convolution

Tags: Convolution

1. We’ll practice some discrete convolutions.
   a. Let $X \sim Ber(p)$ and $Y \sim Ber(q)$ be independent, and $Z = X + Y$. Find $\Omega_Z$ and $p_Z(z)$.
   b. Let $X \sim Bin(n, p)$ and $Y \sim NegBin(r, p)$, and $Z = X + Y$. Let $n = 3, r = 5$. Find $\Omega_Z$ and $p_Z(z)$.

Solution:
   a. We know $\Omega_X = \Omega_Y = \{0, 1\}$, so $\Omega_Z = \{0, 1, 2\}$. By the convolution formula,

   $$p_Z(z) = \sum_{x \in \Omega_X} p_X(x)p_Y(z-x)$$

   We handle them one at a time:

   $$p_Z(0) = p_X(0)p_Y(0) + p_X(1)p_Y(0 - 1) = (1-p)(1-q) + p \cdot 0 = (1-p)(1-q)$$

   $$p_Z(1) = p_X(0)p_Y(1) + p_X(1)p_Y(1 - 1) = (1-p)q + p \cdot (1-q) = q + p - 2pq$$

   $$p_Z(2) = p_X(0)p_Y(2 - 0) + p_X(1)p_Y(2 - 1) = (1-p) \cdot 0 + p \cdot q = pq$$

   Note the two highlighted quantities were 0 because they were out of range!

   b. In our solution here, we will do for the general case of undefined $n$ and $r$.

   We have $\Omega_X = \{0, 1, \ldots, n\}$ and $\Omega_Y = \{r, r+1, \ldots\}$, so $\Omega_Z = \{r, r+1, \ldots\}$.

   We know from the chart that:

   $$p_X(x) = \begin{cases} \binom{n}{x} p^x(1-p)^{n-x}, & k \in \Omega_X \\ 0, & \text{otherwise} \end{cases}$$

   $$p_Y(y) = \begin{cases} \binom{y-1}{r-1} p^r(1-p)^{y-r}, & k \in \Omega_Y \\ 0, & \text{otherwise} \end{cases}$$

   Let $z \in \Omega_Z$. The convolution formula says:

   $$p_Z(z) = \sum_{x \in \Omega_X} p_X(x)p_Y(z-x)$$

   Since we are summing over $\Omega_X$, we don’t need to worry about $p_X(x)$ being 0. We need to ensure that $p_Y(z-x) > 0$ so we need to enforce $z-x \in \Omega_Y \Rightarrow z-x \geq r$ or equivalently $x \leq z - r$. 


We need to handle two cases because if \( z \geq n + r \), then this condition is guaranteed (since \( x \leq n \) (binomial) and \( n \leq z - r \). However, in the case that \( z < n + r \) this is not guaranteed and we have to be careful to make sure that \( x \leq z - r \).

Case 1: If \( r \leq z < n + r \), then

\[
p_Z(z) = \sum_{x=0}^{z-r} p_X(x) p_Y(z-x) = \sum_{x=0}^{z-r} \binom{n}{x} p^x (1-p)^{n-x} \binom{z-x-1}{r-1} p^r (1-p)^{z-x-r}
\]

Case 2: If \( z \geq n + r \), then

\[
p_Z(z) = \sum_{x=0}^{n} p_X(x) p_Y(z-x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \binom{z-x-1}{r-1} p^r (1-p)^{z-x-r}
\]

So our final PMF is

\[
p_Z(z) = \begin{cases} 
\sum_{x=0}^{z-r} \binom{n}{x} p^x (1-p)^{n-x} \binom{z-x-1}{r-1} p^r (1-p)^{z-x-r}, & r \leq z < n + r \\
\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \binom{z-x-1}{r-1} p^r (1-p)^{z-x-r}, & z \geq n + r \\
0, & z < r
\end{cases}
\]
Mitchell and Alex are competing together in a 2-mile relay race. The time Mitchell takes to finish his mile (in hours) is $X \sim \text{Exp} (\lambda = 2)$ and the time Alex takes to finish his mile (in hours) is continuous $Y \sim \text{Unif} (0,1)$. Alex starts immediately after Mitchell finishes his mile, and their performances are independent. What is the distribution of the total time taken to complete this race?

**Solution:**

a. Let $Z = X + Y$ be the total time taken. Then, $\Omega_Z = [0, \infty)$ since $\Omega_X = [0, \infty)$ and $\Omega_Y = [0,1]$. We know from the chart that:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $z \in \Omega_Z$. The convolution formula says:

$$f_Z(z) = \int_{\Omega_Y} f_Y(y) f_X(z - y) dy = \int_0^1 f_Y(y) f_X(z - y) dy$$

Since we are integrating $dy$, we don't need to worry about $f_Y(y)$ being 0. We should check when $f_X(z - y) > 0$. This happens when $z - y \geq 0$ as $\Omega_X = [0, \infty)$. That is, we need $z \geq y$.

We actually need to split into two cases: when $z \in [0,1]$ and $z \in [1, \infty)$: this is because when $z \in [1, \infty)$, we are guaranteed that $z \geq y$ (since $y \in [0,1]$), and when $z \in [0,1]$ we are not.

Case 1: Let $z \in [0,1]$. Then, since we need $y \leq z$:

$$f_Z(z) = \int_0^z f_Y(y) f_X(z - y) dy = \int_0^z 1 \cdot \lambda e^{-\lambda (z - y)} dy = 1 - e^{-\lambda z}$$

Case 2: Let $z \in [1, \infty)$. Then, $y \leq z$ always, so:

$$f_Z(z) = \int_0^1 1 \cdot \lambda e^{-\lambda (z - y)} dy = (e^\lambda - 1) e^{-\lambda z}$$

Hence,

$$f_Z(z) = \begin{cases} 1 - e^{-\lambda z}, & z \in [0,1] \\ (e^\lambda - 1)e^{-\lambda z}, & z \in [1, \infty) \\ 0, & \text{otherwise} \end{cases}$$