Lecture Topics: 5.4 Covariance and Correlation

[Tags: Covariance]

1. The covariance matrix of a random vector \( \mathbf{Z} = (Z_1, Z_2, ..., Z_n) \) is defined to the \( n \times n \) matrix \( \Sigma \) such that \( \Sigma_{ij} = \text{Cov}(X_i, X_j) \). Don’t be too intimidated - it’s just a way to store all the information we need – we won’t be doing any linear algebra with it! The examples will help.

\[
\Sigma = \begin{bmatrix}
\text{Cov}(Z_1, Z_1) = \text{Var}(Z_1) & \text{Cov}(Z_1, Z_2) & \cdots & \text{Cov}(Z_1, Z_n) \\
\text{Cov}(Z_2, Z_1) & \text{Cov}(Z_2, Z_2) = \text{Var}(Z_2) & \cdots & \text{Cov}(Z_2, Z_n) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Cov}(Z_n, Z_1) & \cdots & \cdots & \text{Cov}(Z_n, Z_n) = \text{Var}(Z_n)
\end{bmatrix}
\]

a. Let \( X_1, X_2, X_3, X_4 \) be iid (independent and identically distributed) random variables with mean \( \mu \) and variance \( \sigma^2 \). Let \( \mathbf{X} = (X_1, X_2, X_3, X_4) \) be a random vector with the rvs \( X_i \) as its components. What is the (4 \times 4) covariance matrix of \( \mathbf{X} \)?

b. Define \( \mathbf{Y} = (Y_1, Y_2, Y_3) \) as follows.
   - \( Y_1 = X_1 + X_2 \)
   - \( Y_2 = X_2 + X_3 \)
   - \( Y_3 = X_3 + X_4 \).
   What is the (3 \times 3) covariance matrix of \( \mathbf{Y} \)?

[Tags: Similar to PSet4 Q4, Covariance]

2. Suppose we throw 12 balls independently and uniformly into 7 bins. For \( i = 1, ..., 7 \), let \( X_i \) be the indicator/Bernoulli rv of whether bin \( i \) is empty. Let \( \mathbf{X} = (X_1, ..., X_7) \) be the random vector of indicators.

   a. What is the covariance matrix of \( \mathbf{X} \)?

   b. Let \( Y = \sum_{i=1}^{7} X_i \) be the number of empty bins. What is \( \text{Var}(Y) \)?