CSE 312: Foundations of Computing II
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Lecture Topics: 5.1 Joint Discrete Distributions

[**Tags:** Joint PMFs, Marginal PMFs, Expectation]

1. Suppose we flip a fair coin three times independently. Let X be the number of heads in the first two flips, and Y be the number of heads in the last two flips (there is overlap).
   a. What distribution do X and Y have marginally, and what are their ranges?
   b. What is $p_{X,Y}(x,y)$? Hint: Fill in the margins first representing the marginal distributions!
   c. What is $\Omega_{X,Y}$?
   d. Write a formula for $E[\cos(XY)]$.
   e. Are X and Y independent?

Solution:

a. Marginally, $X, Y \sim Bin\left(2, \frac{1}{2}\right)$ with range $\Omega_X = \Omega_Y = \{0,1,2\}$.
b. I filled in the table first with $(2,2), (0,0), (2,0), (0,2)$ and used the marginal requirements to fill in the rest.

$X \backslash Y$	0	1	2	Σ
0	1/8	1/8	0	1/4
1	1/8	1/4	1/8	1/2
2	0	1/8	1/8	1/4
Σ	1/4	1/2	1/4	1

c. So the joint range is $\Omega_{X,Y} = (\Omega_X \times \Omega_Y) \setminus \{(0,2),(2,0)\}$.
d. $E[\cos(XY)] = \sum_x \sum_y \cos(xy)\, p_{X,Y}(x,y)$.
e. No - $p_X(2) > 0, p_Y(0) > 0$ so $p_X(2) \cdot p_Y(0) > 0$ yet $p_{X,Y}(2,0) = 0$.

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2. Let X be the roll of a fair 3-sided die. We then flip a fair coin X times independently; let Y be the number of heads.
   a. What are Ω_X and Ω_Y? What is $\Omega_{X,Y}$? What is X's marginal distribution?
   b. What is $p_{X,Y}(x,y)$? Hint: Fill in the margins for X!
   c. What is $p_Y(y)$?
   d. Write a formula for $E\left[\frac{X}{Y^2+1}\right]$.
   e. Are X and Y independent?

Solution:

a. We have $\Omega_X = \{1,2,3\}$ and $\Omega_Y = \{0,1,2,3\}$.
b. I filled out this table one row at a time, top to bottom.

$X\backslash Y$	0	1	2	3	Σ
1	1/6	1/6	0	0	1/3
2	1/12	1/6	1/12	0	1/3
3	1/24	1/8	1/8	1/24	1/3
Σ					1

We did solved for each of these by solving:

$P(X = x, Y = y) = P(Y = y \mid X = x)P(X = x) = \binom{x}{y}\left(\frac{1}{2}\right)^x \cdot \frac{1}{3}$

This is because we have $\frac{1}{3}$ chance of getting any of the values for X.

Then whatever we will look for Y heads out of X coin flips.

c. The solving for bottom margin we have:

$X\backslash Y$	0	1	2	3	Σ
1	1/6	1/6	0	0	1/3
2	1/12	1/6	1/12	0	1/3
3	1/24	1/8	1/8	1/24	1/3
Σ	7/24	11/24	5/24	1/24	1

Which in all gives us:

$$p_Y(y) = \begin{cases} 7/24 & y = 0 \\ 11/24 & y = 1 \\ 5/24 & y = 2 \\ 1/24 & y = 3 \end{cases}$$

d. By LOTUS,

$$E\left[\frac{X}{Y^2+1}\right] = \sum_x \sum_y \frac{x}{y^2+1} p_{X,Y}(x,y)$$

e. No again! The joint range isn't the product of the marginals.