1.3.1 Binomial Theorem

The first important idea behind the binomial theorem is the FOIL (first, outer, inner, last) method for distributing two binomials. Recall that:

\[(x + y)^2 = (x + y)(x + y) = (x + y)(x + y) = xx + xy + yx + yy\]

[applying foil and taking the sum of the products of the first terms, outer terms, inner terms, and last terms]

\[= x^2 + 2xy + y^2\]

But, let’s say that we wanted to do this for a binomial raised to some higher power, say \((x + y)^4\). We can use a similar approach.
(x + y)^4 = (x + y)(x + y)(x + y)(x + y) \\
= xxxx + yyyy + xyxy + yxyy + \ldots

But what are the terms exactly that are included in this expression?

Notice that each term will be a mixture of x’s and y’s. In fact, each term will be in the form \(x^ky^{n-k}\) (in this case \(n = 4\)). This is because there will be exactly \(n\) x’s or y’s in each term, so if there are \(k\) x’s, then there must be \(n-k\) y’s.

For a specific \(k\) though, how many times do we get \(x^ky^{n-k}\)? For example, in the above case, take \(k = 1\), then note that \(xyyy = yxyy = yyxy = yyyx = xy^3\), so \(xy^3\) will appear with the coefficient of 4 in the final simplified form (just like for \((x + y)^2\) the term \(xy\) appears with a coefficient 2).

Now, we can generalize this, as the number of terms will simplify to \(x^ky^{n-k}\) will be equivalent to the number of ways to choose exactly \(k\) of the \(n\) binomials to give us \(x\) (and let the remaining \(n-k\) give us \(y\)). To think of this in the above example with \(k = 1\) and \(n = 4\), we were consider which of the four binomials would give us the single \(x\), the first, second, third, or fourth.

Let’s consider \(k = 2\) in the above example. We want to know how many terms are equivalent to \(x^2y^2\). Well, we then have \(xxyy = yxx y = yyxx = xyxy = xyyx = xyxy = x^2y^2\), so there are six ways and the coefficient on the simplified term \(x^2y^2\) will be 6.

Notice that we are essentially choosing which of the binomials gives us an \(x\) such that \(k\) of the \(n\) binomials do. Since the order of choices doesn’t matter after we simplify, we can consider this as \(\binom{n}{k}\). Note that \(\binom{4}{1} = 4\) and \(\binom{4}{2} = 6\), so this aligns with our thinking above.

That leads us to the binomial theorem:

**Theorem 1.3.1.1: Binomial Theorem**

Let \(x, y \in \mathbb{R}\) and \(n \in \mathbb{N}\) a positive integer. Then:

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

This essentially states that in the expansion of the left side, the coefficient of the term with \(x\) raised to the power of \(k\) and \(y\) raised to the power of \(n-k\) will be \(\binom{n}{k}\), and we know this because we are considering the number of ways to choose \(k\) of the \(n\) binomials in the expression to give us \(x\).

This can also be proved by induction, but this is left as an exercise for the reader.

**1.3.2 Inclusion-Exclusion**

Say we did an anonymous survey where we asked whether students in CSE312 like ice cream, and found that 43 people liked ice cream. Then we did another anonymous survey where we asked whether students in CSE312 liked donuts, and found that 20 people liked donuts. With this information can we determine how many people like both ice cream and donuts? Not quite as the 20 people and 43 people could or could not include the same individuals. So, we did another anonymous survey in which we asked whether students
in CSE312 like both ice cream and donuts, and found that 7 people like both. Now, do we have enough information to determine how many students like either ice cream or donuts?

Yes! Knowing that 43 people like ice cream and 7 people like both ice cream and donuts, we can conclude that 36 people like ice cream but don’t like donuts. Similarly, knowing that 20 people like donuts and 7 people like both ice cream and donuts, we can conclude that 13 people like donuts but don’t like ice cream. This leaves us with the following picture, where $A$ is the students who like ice cream. $B$ is the students who like donuts (this implies $|A \cap B|$ is the students who like both):

So we have the following:

\[
|A| = 43 \\
|B| = 20 \\
|A \cap B| = 7
\]

Now, to go back to the question of how many students like either ice cream or donuts, we can just add up the 36 people that just like ice cream, the 7 people that like both ice cream and donuts, and the 13 people that just like donuts, and get $36 + 7 + 13 = 56$. Alternatively, we could consider this as adding up the 43 people who like ice cream (including both the 36 those who just like ice cream and the 7 who like both) and the 20 people who like donuts (including the 13 who just like donuts and the 7 who like both) and then subtract the 7 who like both since they were counted twice. That is $43 + 20 - 7 = 56$. That leaves us with:

\[
|A \cup B| = 36 + 7 + 13 = 56 = 43 + 20 - 7 = |A| + |B| - |A \cap B|
\]

Recall that $|A \cup B|$ is the students who like either donuts or ice cream (the union of the two sets).
Theorem 1.3.2.1: Inclusion-Exclusion

Let $A, B$ be sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Further, in general, if $A_1, A_2, \ldots, A_n$ are sets, then:

$$|A_1 \cup \cdots \cup A_n| = \text{singles} - \text{doubles} + \text{triples} - \text{quads} + \cdots$$

$$= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|)$$
$$+ (A_1 \cap A_2 \cap A_3) + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| + \cdots$$

Where singles are the size of all the single sets, doubles are the size of all the intersections of two sets, triples are the size of all the intersections of three sets, quads are all the intersection of four sets, and so forth.

1.3.3 Pigeonhole Principle

Consider the question of if 11 children have to share 3 beds, at least one bed must contain at least how many children? What this is asking is what is the largest number that if all 11 children are put to bed, that at least one bed must be guaranteed to hold?

Well we can consider this by trying different arrangements, such as 1 child in the first bed, 1 child in the second bed, 9 children in the last bed but that’s not too helpful as we could just rearrange this to 2 children in the first bed, 1 child in the second bed, 8 children in the last bed and now we know we can’t be guaranteed anything about 8 or 9.

A better approach would be to first distribute the children evenly amongst the beds, say put 3 children in each bed to start. That leaves us with 3 times 3 equals 9 children accounted for, and 2 children remaining with a bed. Well, they must be put to bed, so we can put each of them in a separate bed and we finish with the first bed having 4, the second bed having 4, and the third bed having 3. No matter how we move the children around, we can’t have an an arrangement where at least one bed will contain at least 4 children.

We could also have found this by dividing 11 by 3 and rounding up to account for the remainder.

Before formally defining the pigeon hole principle, consider the floor and ceiling functions.

**Definition 1.3.3.1: Floor and Ceiling Functions**

The floor function $\lfloor x \rfloor$ rounds down.

The ceiling function $\lceil x \rceil$ rounds up.

**Examples**

$\lfloor 2.5 \rfloor = 2$  
$\lfloor 2.5 \rfloor = 3$  
$\lceil 16.999999 \rceil = 16$  
$\lceil 9.000301 \rceil = 10$  
$\lceil 5 \rceil = 5$
**Theorem 1.3.3.1: Pigeonhole Principle**

If there are \( n \) pigeons we want to put into \( k \) holes (where \( n > k \)), then at least one pigeonhole must contain at least 2 pigeons.

More generally, if there are \( n \) pigeons we want to put into \( k \) pigeonholds, then at least one pigeonhold must contain at least \( \lceil n/k \rceil \) pigeons.

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### 1.3.4 Combinatorial Proofs

Suppose we wanted to show that \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

We could start with an algebraic approach and try something like:

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)}{(k-1)!(n-k)!} + \frac{(n-1)}{k!(n-1-k)!} \\
\cdots \\
= \frac{n!}{k!(n-k)!} \\
= \binom{n}{k}
\]

However, those \( \cdots \) are nontrivial and take a lot of algebra we don’t want to do.

So, let’s consider another approach. Maybe we can describe how both the left and the right side count the same thing in different ways.

In this case, let’s consider the set of numbers \( \{1, 2, \ldots n\} \).

Then the left side, \( \binom{n}{k} \) is just the number of subsets of size \( k \), by definition.

Then the right size is the same quantity broken down into two cases. The first case is including 1 in the each of the subsets of size \( k \). Then we need to choose \( k-1 \) of the remaining \( n-1 \) numbers (\( n \) numbers excluding 1 is \( n-1 \) numbers) to make a subset of size \( k \) which includes 1. There are \( \binom{n-1}{k-1} \) ways to do this. The second case is not including 1 in the subsets of size \( k \). Then we need to choose \( k \) numbers from the remaining \( n-1 \) numbers. There are \( \binom{n-1}{k} \) ways to do this. So, in total we have \( \binom{n-1}{k-1} + \binom{n-1}{k} \) possible subsets.

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**Definition 1.3.4.1: Combinatorial Proofs**

To prove two quantities are equal, you can come up with a combinatorial situation, and show that both in fact count the same thing, and hence must be equal.

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### 1.3.5 Exercises

1. Calculate the coefficient of \( a^{45}b^{14} \) in the expansion \( (4a^3 - 5b^2)^{22} \).
Solution: Let \( x = 4a^3 \) and \( y = -5b^2 \). Then, we are looking for the coefficient of \( x^{15}y^7 \), which is \( \binom{22}{15} \). So we have the term

\[
\binom{22}{15} x^{15}y^7 = \binom{22}{15} (4a^3)^{15}(-5b^2)^7 = \left( -\binom{22}{15} 4^{15}5^7 \right) a^{45}b^{14}
\]

and our answer is \( -\binom{22}{15} 4^{15}5^7 \).

2. How many numbers in the set \([360] = \{1, 2, \ldots, 360\}\) are divisible by:

(a) 4, 6, and 9.
(b) 4, 6 or 9.
(c) neither 4, 6, nor 9.

Solution:

(a) This is just the multiplies of lcm(4, 6, 9) = 36, which there are \( \frac{360}{36} = 10 \) of.

(b) Let \( D_i \) be the number of numbers in \([360]\) which are divisible by \( i \), for \( i = 4, 6, 9 \). Then, by inclusion-exclusion,

\[
|D_4 \cup D_6 \cup D_9| = |D_4| + |D_6| + |D_9| - |D_4 \cap D_6| - |D_4 \cap D_9| - |D_6 \cap D_9| + |D_4 \cap D_6 \cap D_9|
\]

\[
= \frac{360}{4} + \frac{360}{6} + \frac{360}{9} - \frac{360}{12} - \frac{360}{36} - \frac{360}{18} + \frac{360}{36}
\]

Notice the denominators for the paired terms are again, dividing by the least common multiple.

(c) Complementary counting - this is just 360 minus the answer from the previous part.

3. How many cards must you draw from a standard 52-card deck (4 suits and 13 cards of each suit) until you are guaranteed to have:

(a) A single pair? (e.g., AA, 99, JJ)
(b) Two (different) pairs? (e.g., AAKK, 9933, 44QQ)
(c) A full house (a triple and a pair)? (e.g., AAAKK, 99922, 555JJ)
(d) A straight (5 in a row, with the lowest being A,2,3,4,5 and the highest being 10,J,Q,K,A)?
(e) A flush (5 cards of the same suit)? (e.g., 5 hearts, 5 diamonds)
(f) A straight flush (5 cards which are both a straight and a flush)?

Solution:

(a) The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. So the answer is 14.

(b) The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. But then we could draw the other two of that pair as well to get 16 still without two pairs. So the answer is 17.

(c) The worst that could happen is to draw all pairs (26 cards). Then the next is guaranteed to cause a triple. So the answer is 27.

(d) The worst that could happen is to draw all the A - 4, 6 - 9, and J - K. After drawing these \( 11 \cdot 4 = 44 \) cards, we could still fail to have a straight. Finally, getting a 5 or 10 would give us a straight. So the answer is 45.

(e) The worst that could happen is to draw 4 of each suit (16 cards), and still not have a flush. So the answer is 17.
4. Show that in a group of \( n \) people (who may be friends with any number of other people), two must have the same number of friends.

**Solution:** We have two cases.

(a) Case 1: Everyone has at least one friend. Then, everyone has a number of friends between 1, 2, \ldots, \( n-1 \). By the pigeonhole principle, since there are \( n \) people and \( n-1 \) possibilities, at least two people have the same number of friends.

(b) Case 2: At least one person has no friends. Let’s take one such person and call them A. Then, the other \( n-1 \) people can have number of friends from 0, \ldots, \( n-2 \) since they can’t be friends with A. We have two more cases within this case unfortunately.

i. Case 2a: If one of these \( n-1 \) people has no friends, we are done since A and this person both have 0 friends.

ii. Case 2b: Otherwise, these people all have at least one friend, from 1, \ldots, \( n-2 \), and since there are \( n-1 \) people and \( n-2 \) possibilities, at least two people have the same number of friends.

In all cases we are guaranteed that two people have the same number of friends.

5. Prove the following two identities combinatorially (NOT algebraically):

(a) Prove that \( \binom{n}{m} \binom{m}{k} = \binom{n}{m} \binom{n-m}{k-m} \).

(b) Prove that \( 2^n = \sum_{k=0}^{n} \binom{n}{k} \).

**Solution:**

(a) We’ll show that both sides count, from a group of \( n \) people, the number of committees of size \( m \), and within that committee a subcommittee of size \( k \).

**Left-hand side:** We first choose \( m \) people to be on the committee from \( n \) total; there are \( \binom{n}{m} \) ways to do so. Then, within those \( m \), we choose \( k \) to be on a specialized subcommittee; there are \( \binom{m}{k} \) ways to do so. By the product rule, the number of ways to assign these is \( \binom{n}{m} \binom{m}{k} \).

**Right-hand side:** We first choose which \( k \) to be on the subcommittee of size \( k \); there are \( \binom{n}{k} \) ways to do so. From the remaining \( n-k \) people, we choose \( m-k \) to be on the committee (but not the subcommittee). By the product rule, the number of ways to assign these is \( \binom{n}{k} \binom{n-k}{m-k} \).

Since the LHS and RHS both count the same thing, they must be equal.

(b) We’ll argue that both sides count the number of subsets of the set \( [n] = \{1, 2, \ldots, n\} \).

**Left-hand side:** Each element we can have in our subset or not. For the first element, we have 2 choices (in or out). For the second element, we also have 2 choices (in or out). And so on. So the number of subsets is \( 2^n \).

**Right-hand side:** The subset can be of any size ranging from 0 to \( n \), so we have a sum. Now how many subsets are there of size exactly \( k \)? There are \( \binom{n}{k} \) because we choose \( k \) out of \( n \) to have in our set (and order doesn’t matter in sets)! Hence, the number of subsets is \( \sum_{k=0}^{n} \binom{n}{k} \).

Since the LHS and RHS both count the same thing, they must be equal.
It's cool to note we can also prove this with the binomial theorem setting $x = 1$ and $y = 1$ - try this out! It takes just one line!