0.3.1 Summation Notation

Suppose that we want to write the sum: 1 + 2 + 3 + 5 + 6 + 7 + 8 + 9 + 10. We can write out each element, but it becomes tedious. We could use dots, to signify this as: 1 + 2 + ··· + 9 + 10, but this can become vague. Instead, we can use summation notation as shorthand for summations of values. Here we are referring to the sum of each element \( i \), where \( i \) will take on every value in the range starting with 1 and ending with 10.

\[
1 + 2 + 3 + \cdots + 10 = \sum_{i=1}^{10} i
\]

Further, we can use equations in these definitions. In the below, in the first equation (0.3.1) \( j \) takes on the values 4 to 9 and the square of each of these values will be summed together. Note that this is equivalent to \( k \) taking on the values of 1 to 6, and adding 3 to each of the values before squaring and summing them up (0.3.2).

\[
16 + 25 + 36 + \cdots + 81 = \sum_{j=4}^{9} j^2 \tag{0.3.1}
\]
\[
= \sum_{k=1}^{6} (k + 3)^2 \tag{0.3.2}
\]

This brings us to the following definition of summation notation:

**Definition 0.3.1.1: Summation Notation**

Let \( x_1, x_2, x_3, \ldots \) be a sequence of numbers. Then, the following notation represents the sum \( x_a + x_{a+1} + \cdots + x_{b-1} + x_b \):

\[
\sum_{i=a}^{b} x_i
\]
Further, if \( S \) is a set, and \( f : S \rightarrow \mathbb{R} \) is a function defined on \( S \), then the following notation sums over all elements \( x \in S \) of \( f(x) \):

\[
\sum_{x \in S} f(x)
\]

Note that the sum over no terms is defined as 0.

Further, the associative and distributive properties hold for sums.

**Fact 0.3.1.1: The Associative and Distributive Properties of Sums**

We have the associative property (0.3.3) and distributive property (0.3.4, 0.3.5) for sums.

\[
\sum_{x \in A} f(x) + \sum_{x \in A} g(x) = \sum_{x \in A} (f(x) + g(x)) \quad (0.3.3)
\]

\[
\sum_{x \in A} \alpha \cdot f(x) = \alpha \sum_{x \in A} (f(x)) \quad (0.3.4)
\]

\[
(\sum_{x \in A} f(x))(\sum_{y \in B} g(x)) = \sum_{x \in A} \sum_{y \in B} f(x)g(x) \quad (0.3.5)
\]

The proof of this is left to the reader, but see the examples below for some intuition!

**Examples**

For, \( \sum_{k=3}^{7} k^{10} \), we raise each value from 3 to 7 to the power of 10 and sum them together. That is:

\[
\sum_{k=3}^{7} k^{10} = 3^{10} + 4^{10} + 5^{10} + 6^{10} + 7^{10}
\]

Then if, we let \( S = \{3, 6, 8, 11\} \), for \( \sum_{y \in S} (2^{y} + 5) \), raise 2 to the power of each value in \( S \) and sum the results together. That is

\[
\sum_{y \in S} (2^{y} + 5) = (2^{3} + 5) + (2^{6} + 5) + (2^{8} + 5) + (2^{11} + 5)
\]

For the sum of a constant, \( \sum_{t=6}^{8} 4 \), we add the constant, 4 for each of the value in the range.

\[
\sum_{t=6}^{8} 4 = 4 + 4 + 4 + 4
\]

Finally, for a range with no values, the sum is defined as 0, for \( \sum_{z=2}^{1} \sin(z) \), where there are no values from 2 to 1, we have:

\[
\sum_{z=2}^{1} \sin(z) = 0
\]
Looking at the associative property, consider the following:

\[
\sum_{i=5}^{6} i + \sum_{i=5}^{6} i^2 = (5 + 6 + 7) + (5^2 + 6^2 + 7^2) = (5 + 5^2) + (6 + 6^2) + (7 + 7^2) = \sum_{i=5}^{6} i + i^2
\]

Also, for the distributive property consider:

\[
\sum_{i=3}^{5} 2i = 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 = 2(3 + 4 + 5) = 2 \sum_{i=3}^{5} i
\]

and:

\[
\prod_{i=1}^{2} f(a_i) \prod_{j=1}^{3} g(b_j) = (f(a_1) + f(a_2))(g(b_1) + g(b_2) + g(b_3)) = (f(a_1)g(b_1) + f(a_1)g(b_2) + f(a_1)g(b_3) + f(a_2)g(b_1) + f(a_2)g(b_2) + f(a_2)g(b_3)) = \sum_{i=1}^{2} \sum_{j=1}^{3} f(a_i)g(b_j)
\]

### 0.3.2 Product Notation

Similarly, we can define product notation to handle multiplications.

**Definition 0.3.2.1: Product Notation**

Let \(x_1, x_2, x_3, \ldots\) be a sequence of numbers. Then, the following notation represents the sum \(x_a \cdot x_{a+1} \cdot \cdots \cdot x_{b-1} \cdot x_b:\)

\[
\prod_{i=a}^{b} x_i
\]

Further, if \(S\) is a set, and \(f : S \rightarrow \mathbb{R}\) is a function defined on \(S\), then the following notation multiplies over all elements \(x \in S\) of \(f(x)\):

\[
\prod_{x \in S} f(x)
\]

Note that the product over no terms is defined as 1.

**Examples**

For \(\prod_{a=4}^{7}\), we multiply each value in the range 4 to 7 and have:

\[
\prod_{a=4}^{7} = 4 \cdot 5 \cdot 6 \cdot 7
\]
Then if, we let $S = \{3, 6, 8, 11\}$, for $\prod_{x \in S} 8$, we multiply 8 for each value in the set, $S$ and have:

$$\prod_{x \in S} 8 = 8 \cdot 8 \cdot 8 \cdot 8$$

Then for $\prod_{z=2}^{1} \sin(z)$, we have the empty product, because there are no values in the range 2 to 1, so we have:

$$\prod_{z=2}^{1} \sin(z) = 1$$