Markov Chains

~~Two~~ ONE applications

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Some slides by Alex Tsun

Agenda

- Discrete-Time Stochastic Processes
- Review Markov Chains
- Pagerank
- ~~The metropolis algorithm~~

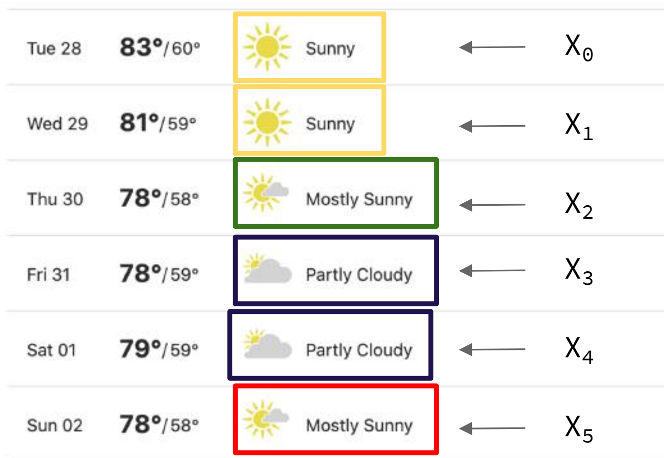
Discrete-Time Stochastic Processes

Discrete-Time Stochastic Process (DTSP): A discrete-time stochastic process is a sequence of random variables $X_0, X_1, X_2, \ldots$ where X_t is the value at time t.

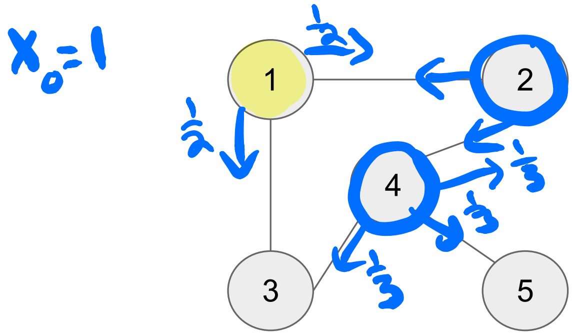
Examples:

- The temperature in Seattle each day.
- The stock price of TESLA each day.
- Your position on a one-dimensional number line during a "random walk".
- The Bernoulli process (at each time step, flip a coin independently).

Discrete-Time Stochastic Processes

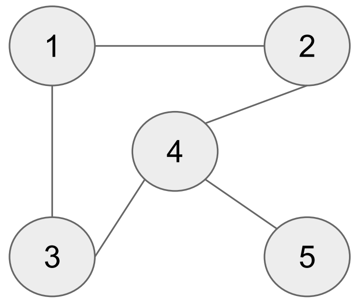


Random Walk on a Graph Example



Suppose we start at node 1, and at each time step, independently step to a neighboring node with equal probability.

Random Walk on a Graph Example

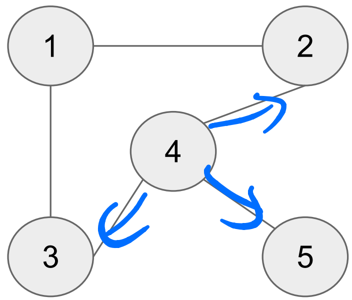


Suppose we start at node 1, and at each time step, independently step to a neighboring node with equal probability.

For example, $X_0 = 1$ since at time $t = 0$ we are at node 1. Then, X_1 is either 2 or 3 (but cannot be 4 or 5 since not neighbors of node 1). And so on.

This DTSP is actually a special type of DTSP called a **Markov Chain**.

Random Walk on a Graph Example



Three Key Properties:

1. We only have finitely many states (we have 5 in this example: $\{1,2,3,4,5\}$).
2. We don't care about the past, given the present. That is, the distribution of where we go next **ONLY** depends on where we are ***currently***, and not any past history.
3. The transition probabilities are the same at each time step (e.g., if we are at node 1 at time $t = 0$ or $t = 152$, we always are equally likely to go to node 2 or 3).

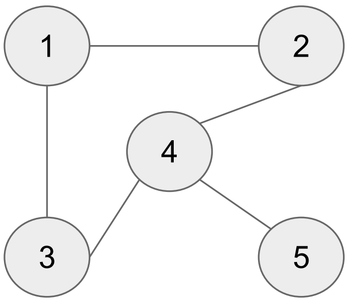
Markov Chains

Discrete time stochastic process $X_1, X_2, \dots, X_t, \dots$
Such that

$S = \{1, 2, \dots, n\}$

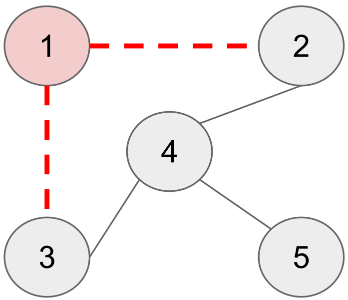
- There is a state space S, such that each X_t is in S.
- Pr $(X_{t+1}$ = j | X_t = i, history) = Pr $(X_{t+1}$ = j | X_t = i)
- p_{ij} = Pr $(X_{t+1}$ = j | X_t = i) for all i and j specified by transition probability matrix (TPM) P

Transition Probability Matrix Example



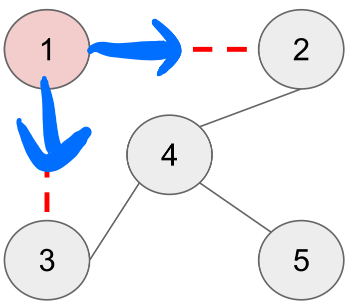
	To s_1	To s_2	To s_3	To s_4	To s_5
From s_1					
From s_2					
From s_3					
From s_4					
From s_5					

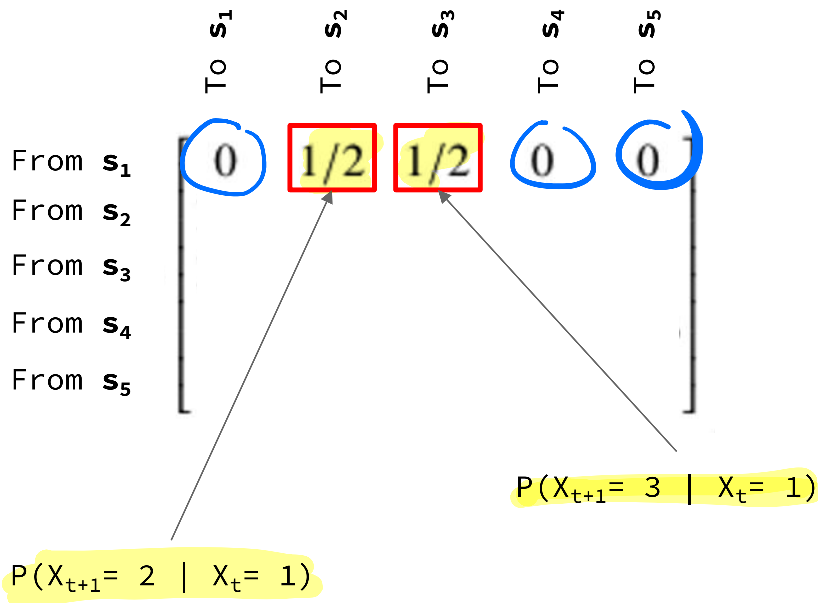
Transition Probability Matrix Example



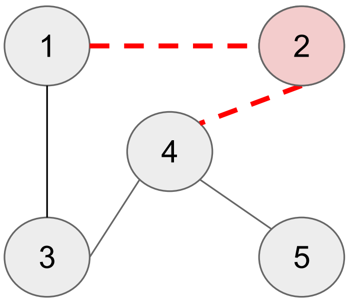
	To s_1	To s_2	To s_3	To s_4	To s_5
From s_1					
From s_2					
From s_3					
From s_4					
From s_5					

Transition Probability Matrix Example





Transition Probability Matrix Example

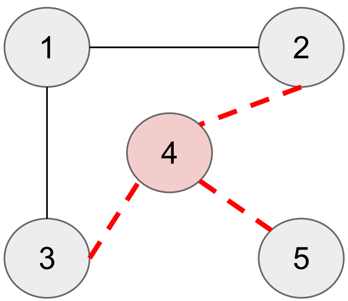


	To s_1	To s_2	To s_3	To s_4	To s_5
From s_1	0	1/2	1/2	0	0
From s_2	1/2	0	0	1/2	0
From s_3					
From s_4					
From s_5					

$P(X_{t+1} = 4 \mid X_t = 2)$

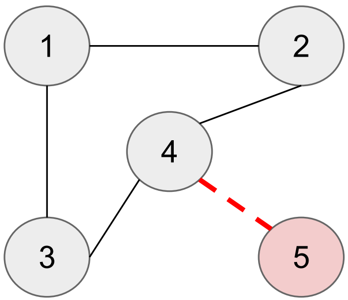
$P(X_{t+1} = 1 \mid X_t = 2)$

Transition Probability Matrix Example



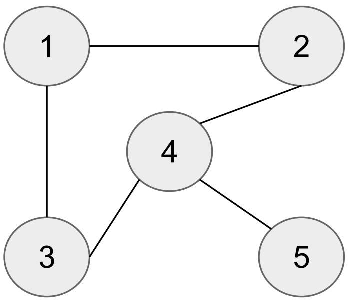
	To s_1	To s_2	To s_3	To s_4	To s_5
From s_1	0	1/2	1/2	0	0
From s_2	1/2	0	0	1/2	0
From s_3	1/2	0	0	1/2	0
From s_4	0	1/3	1/3	0	1/3
From s_5					

Transition Probability Matrix Example



	To s_1	To s_2	To s_3	To s_4	To s_5
From s_1	0	1/2	1/2	0	0
From s_2	1/2	0	0	1/2	0
From s_3	1/2	0	0	1/2	0
From s_4	0	1/3	1/3	0	1/3
From s_5	0	0	0	1	0

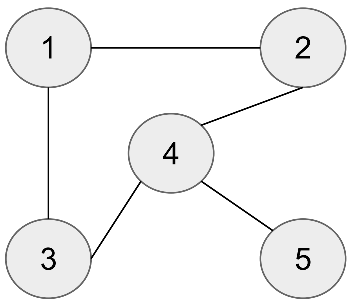
Transition Probability Matrix Example



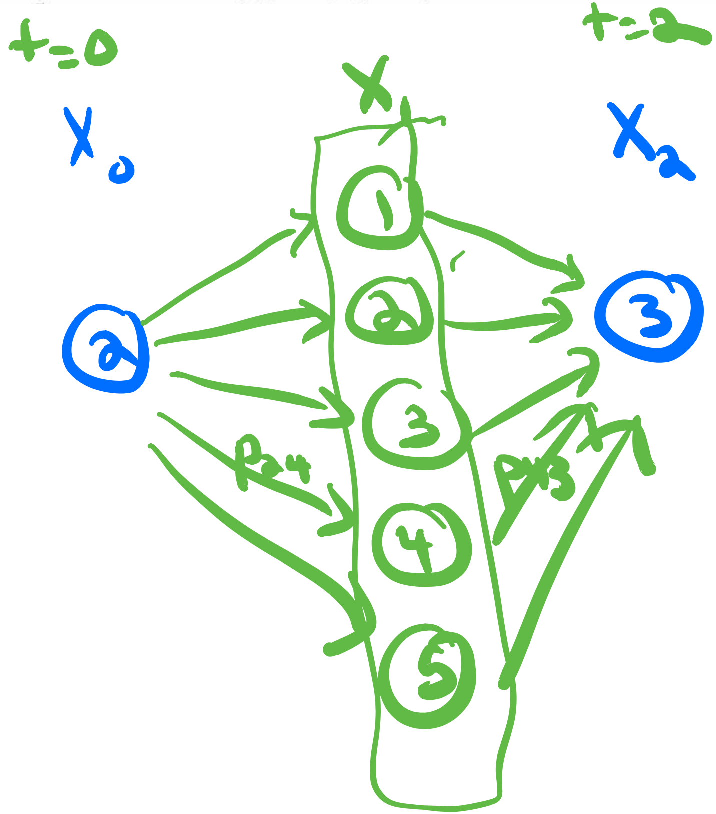
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Computing Probability Example

$P(X_2 = 3 | X_0 = 2)$



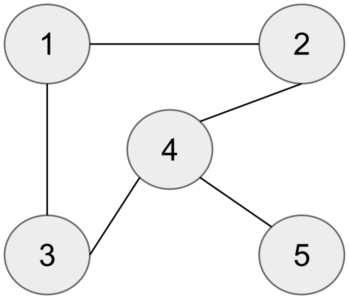
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Computing Probability Example

in state 3 the 2

2 at the 0



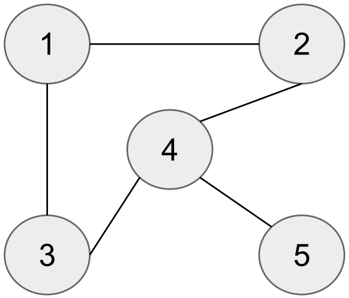
$$P(X_2 = 3 | X_0 = 2)$$

$$= \sum_{i=1}^{5} P(X_2 = 3 | X_0 = 2, X_1 = i) P(X_1 = i | X_0 = 2) \text{ [LTP]}$$

$$P(X_2 = 3 | X_1 = i)$$

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Computing Probability Example



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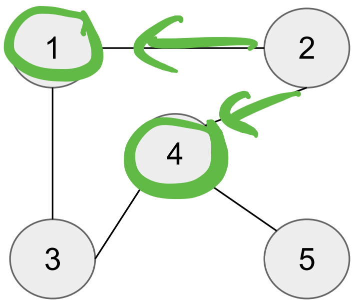
$$= \sum_{i=1}^{5} P(X_2 = 3 | X_1 = i) P(X_1 = i | X_0 = 2) \text{ [Markov property]}$$

P_{i3} P_{2i}

$P_{2i} \cdot P_{i3}$

Computing Probability Example



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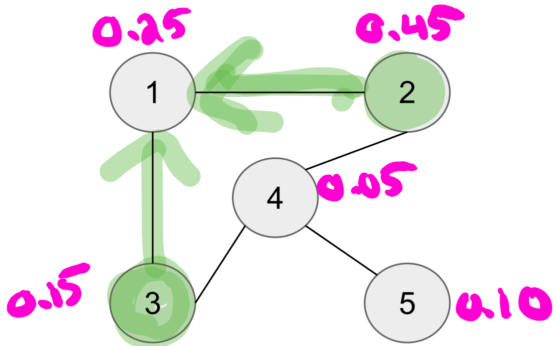
$$= \sum_{i=1}^{5} P(X_2 = 3 | X_1 = i) P(X_1 = i | X_0 = 2) \text{ [Markov property]}$$

$$= P(X_2 = 3 | X_1 = 1) P(X_1 = 1 | X_0 = 2) + P(X_2 = 3 | X_1 = 4) P(X_1 = 4 | X_0 = 2)$$

$$= P_{13} P_{21} + P_{43} P_{24} = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{12}$$

Random Picture

Applying a Transition Matrix



Suppose we start at a random state with these probabilities (sum to 1).

$$v = [0.25, \quad 0.45, \quad 0.15, \quad 0.05, \quad 0.10]$$

$P(X_0 = 1)$ $P(X_0 = 2)$ $P(X_0 = 3)$ $P(X_0 = 4)$ $P(X_0 = 5)$

What is my belief distribution for where I am next?

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$Pr(X_1 = 1)$

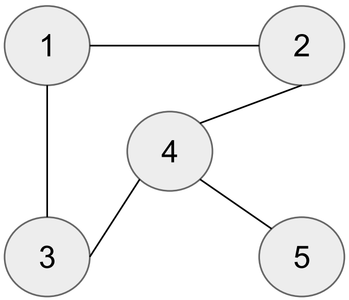
a) 0.25

b) 1

c) $0.45 \cdot \frac{1}{2} + 0.15 \cdot \frac{1}{2}$

d) $\sum_{i=1}^{5} Pr(X_0 = i) Pr(X_1 = 1 | X_0 = i)$

Applying a Transition Matrix



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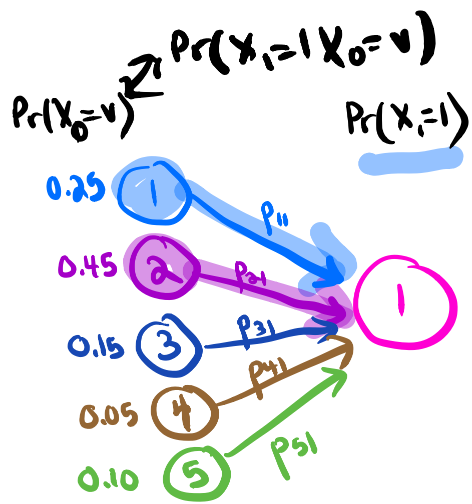


Applying a Transition Matrix

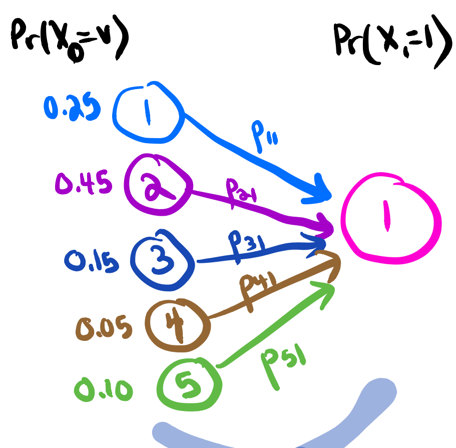
Let's compute the matrix product vP (we'll see why soon!).

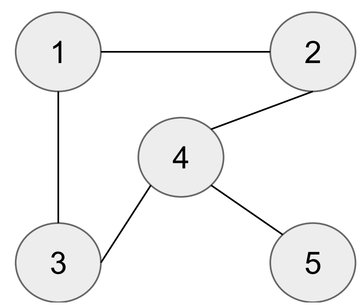
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$P(X_0 = 1)$ $P(X_0 = 2)$ $P(X_0 = 3)$ $P(X_0 = 4)$ $P(X_0 = 5)$

$$v = [0.25, \quad 0.45, \quad 0.15, \quad 0.05, \quad 0.10]$$

$$\left[\sum_{i=1}^{5} P_{i1} v_i , \qquad \right]$$





Applying A Transition Matrix

Let's compute the matrix product vP (we'll see why soon!).

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$P(X_0 = 1)$ $P(X_0 = 2)$ $P(X_0 = 3)$ $P(X_0 = 4)$ $P(X_0 = 5)$

$$v = [0.25, \quad 0.45, \quad 0.15, \quad 0.05, \quad 0.10]$$

$$\left[\sum_{i=1}^{5} P_{i1} v_i, \quad \right]$$

$P(X_1 = 1)$

$$\sum_{i=1}^{5} P_{i1} v_i = \sum_{i=1}^{5} P(X_1 = 1 | X_0 = i) P(X_0 = i) = P(X_1 = 1)$$

Def of TPM (works for any time) | Def of v | LTP

Applying a Transition Matrix

Let's compute the matrix product vP (we'll see why soon!).

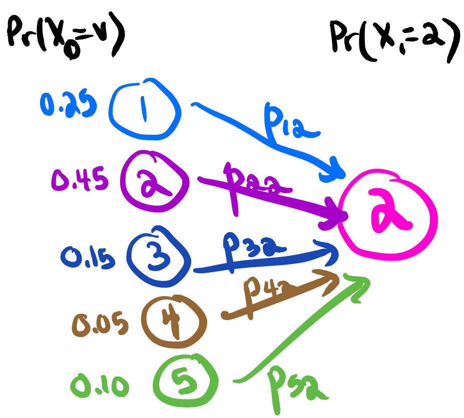
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

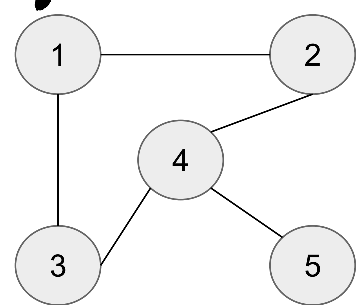
$P(X_0 = 1)$ $P(X_0 = 2)$ $P(X_0 = 3)$ $P(X_0 = 4)$ $P(X_0 = 5)$

$$v = [0.25, \quad 0.45, \quad 0.15, \quad 0.05, \quad 0.10]$$

$$\left[\sum_{i=1}^{5} P_{i1} v_i, \quad \sum_{i=1}^{5} P_{i2} v_i, \quad \right]$$

$P(X_1 = 1)$ $Pr(X_1=2)$





Applying a Transition Matrix

Let's compute the matrix product vP (we'll see why soon!).

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$P(X_0 = 1)$ $P(X_0 = 2)$ $P(X_0 = 3)$ $P(X_0 = 4)$ $P(X_0 = 5)$

$$v = [0.25, \quad 0.45, \quad 0.15, \quad 0.05, \quad 0.10]$$

$$\left[\sum_{i=1}^{5} P_{i1} v_i, \quad \sum_{i=1}^{5} P_{i2} v_i, \quad \right]$$

$P(X_1 = 1)$ $P(X_1 = 2)$

$$\sum_{i=1}^{5} P_{i2} v_i = \sum_{i=1}^{5} P(X_1 = 2 | X_0 = i) P(X_0 = i) = P(X_1 = 2)$$

Def of TPM (works for any time)

Def of v

LTP

Applying A Transition Matrix

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Let's compute the matrix product vP (we'll see why soon!).

$P(X_0 = 1)$ $P(X_0 = 2)$ $P(X_0 = 3)$ $P(X_0 =$

$v = [0.25, \quad 0.45, \quad 0.15,$

HMMMM

$P_{i2}v_i,$ $]$

$= 2)$

$$\sum_{i=1}^{5} P_{i2}v_i = \sum_{i=1}^{5} P(X_1 = 2 | X_0 = i) P(X_0 = i) = P(X_1 = 2)$$

Def of TPM (works for any time) | Def of v | LTP

Applying A Transition Matrix

Let's compute the matrix product vP (we'll see why soon!).

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$P(X_0 = 1)$ $P(X_0 = 2)$ $P(X_0 = 3)$ $P(X_0 = 4)$ $P(X_0 = 5)$

$$v = [0.25, \quad 0.45, \quad 0.15, \quad 0.05, \quad 0.10]$$

$$\left[\sum_{i=1}^{5} P_{i1}v_i, \quad \sum_{i=1}^{5} P_{i2}v_i, \quad \sum_{i=1}^{5} P_{i3}v_i, \quad \sum_{i=1}^{5} P_{i4}v_i, \quad \sum_{i=1}^{5} P_{i5}v_i\right]$$

$P(X_1 = 1)$ $P(X_1 = 2)$ $P(X_1 = 3)$ $P(X_1 = 4)$ $P(X_1 = 5)$

Right-multiplying by P gives the belief distribution at the next time step!

$$vP = (P(X_1=1), Pr(X_1=2), Pr(X_1=3), P(X_1=4), P(X_1=5)) \cdot P$$

$$vP^2 = (P(X_2=1), Pr(X_2=2), \ldots)$$

$$vP^t = (P(X_t=1), Pr(X_t=2), \ldots)$$

Applying a Transition Matrix

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Let's compute the matrix product vP (we'll see why soon!).

$P(X_0= 1)$	$P(X_0= 2)$	$P(X_0= 3)$	$P(X_0= 4)$	$P(X_0= 5)$

$$v = [0.25, \quad 0.45, \quad 0.15, \quad 0.05, \quad 0.10]$$

$$(1, 0, 0, 0, 0)$$

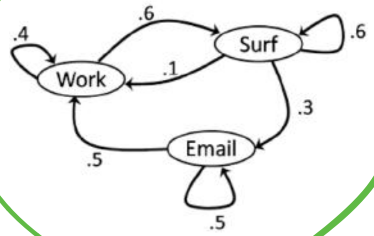
$$\left[\sum_{i=1}^{5} P_{i1}v_i, \quad \sum_{i=1}^{5} P_{i2}v_i, \quad \sum_{i=1}^{5} P_{i3}v_i, \quad \sum_{i=1}^{5} P_{i4}v_i, \quad \sum_{i=1}^{5} P_{i5}v_i\right]$$

$P(X_1= 1)$	$P(X_1= 2)$	$P(X_1= 3)$	$P(X_1= 4)$	$P(X_1= 5)$

Right-multiplying by P gives the belief distribution at the next time step!

By induction (repeatedly applying this matrix P), vP^t is the distribution after t time steps.

$$vP^t = (Pr(X_t=1), \ldots)$$



$$P^2 = \begin{matrix} & W & S & E \\ W & .22 & .6 & .18 \\ S & .25 & .42 & .33 \\ E & .45 & .3 & .25 \end{matrix}$$

$$P^3 = \begin{matrix} & W & S & E \\ W & .238 & .492 & .270 \\ S & .307 & .402 & .291 \\ E & .335 & .450 & .215 \end{matrix}$$

$$P^{10} \approx \begin{matrix} & W & S & E \\ W & .2940 & .4413 & .2648 \\ S & .2942 & .4411 & .2648 \\ E & .2942 & .4413 & .2648 \end{matrix}$$

$$P^{60}_{11} + P^{60}_{12} + P^{60}_{13} = 1$$

$$P^{30} \approx \begin{matrix} & W & S & E \\ W & .29411764705 & .44117647059 & .26470588235 \\ S & .29411764706 & .44117647058 & .26470588235 \\ E & .29411764706 & .44117647059 & .26470588235 \end{matrix}$$

$$P^{60} \approx \begin{matrix} & W & S & E \\ W & .294117647058823 & .441176470588235 & .264705882352941 \\ S & .294117647068823 & .441176470588235 & .264705882352941 \\ E & .294117647068823 & .441176470588235 & .264705882352941 \end{matrix}$$

α_1 α_2 α_3

$v = (0.51, 0.32, 0.17)$

vP^{60}

$0.51 + 0.32 + 0.17 = 1$

$0.51\alpha_1 + 0.32\alpha_1 + 0.17\alpha_1$

$v P^{60} = (\alpha_1, \alpha_2, \alpha_3)$

$= \alpha_1$

Applying a Transition Matrix

$P^t =$ for t large

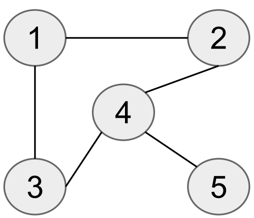
$$\begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.3 & 0.1 \end{bmatrix}$$

P(X$_0$= 1) P(X$_0$= 2) P(X$_0$= 3) P(X$_0$= 4) P(X$_0$= 5)

$$v = [0.25, \quad 0.45, \quad 0.15, \quad 0.05, \quad 0.10]$$

$$\left[\sum_{i=1}^{5} P_{i1} v_i, \quad \sum_{i=1}^{5} P_{i2} v_i, \quad \sum_{i=1}^{5} P_{i3} v_i, \quad \sum_{i=1}^{5} P_{i4} v_i, \quad \sum_{i=1}^{5} P_{i5} v_i\right]$$

P(X$_1$= 1) P(X$_1$= 2) P(X$_1$= 3) P(X$_1$= 4) P(X$_1$= 5)



$v P^t = (0.2 \; 0.2 \; 0.2 \; 0.3 \; 0.1)$

Applying a Transition Matrix

Let's compute the matrix product
vP (we'll see why soon!).

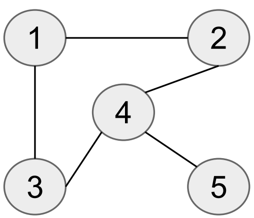
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$P(X_t = 1)$	$P(X_t = 2)$	$P(X_t = 3)$	$P(X_t = 4)$	$P(X_t = 5)$
(0.2	0.2	0.2	0.3	0.1.)

$=\pi$

$$\left[\sum_{i=1}^{5} P_{i1}v_i, \quad \sum_{i=1}^{5} P_{i2}v_i, \quad \sum_{i=1}^{5} P_{i3}v_i, \quad \sum_{i=1}^{5} P_{i4}v_i, \quad \sum_{i=1}^{5} P_{i5}v_i\right]$$

$P(X_1 = 1)$	$P(X_1 = 2)$	$P(X_1 = 3)$	$P(X_1 = 4)$	$P(X_1 = 5)$



$\pi = (0.2, 0.2, 0.2, 0.3, 0.1)$

vP^t as t gets large

$\pi P = \pi$

Stationary Distribution of a Markov Chain

Stationary Distribution: The stationary distribution of a Markov Chain with n states (which doesn't always exist), is the n-dimensional row vector π (which must be a probability distribution – nonnegative and sums to 1), such that

$$\pi P = \pi$$

Intuitively, it means that the distribution at the next time step is the same as the distribution at the current time step. This typically happens after a "long time" (called the **mixing time**) in the process (meaning after lots of transitions are taken).

$$(\pi_1, \pi_2, \ldots, \pi_n)$$

Stationary Distribution of a Markov Chain

Fundamental Theorem of Markov Chains

Let P be the TPM for a Markov chain. Under minor technical conditions, including "connectivity"

$$\lim_{t\to\infty} P_{ij}^t = \pi_j \qquad \forall i, j$$

where

$$\pi_j = \sum_{i=1}^{n} \pi_i p_{ij} \qquad j = 1, \ldots, n$$

$$\pi_1 + \pi_2 + \ldots + \pi_n = 1$$

π is stationary

$\pi = \pi P$

No more CCs.

Application 1: PageRank

1997

- Bill Clinton in White House
- Deep Blue beat world chess champion (Kasparov)

And the Internet kind of sucked

Nov ‘97: only one of the top 4 commercial search engines actually *found itself* when you search

The problem

Search engines worked by matching words

Top search for Bill Clinton

- `Bill Clinton Joke of the Day' Website

Deeply susceptible to spammers and advertisers

How to fix?

Collect pages with decent textual match

Then **rank** them by some measure of ‘quality’ or ‘authority’.

Enter two groups:

- Jon Kleinberg (prof at Cornell)
- Larry Page and Sergey Brin (Ph.D. students at Stanford)

Both groups had the same brilliant idea

Larry Page and Sergey Brin (Ph.D. students at Stanford)

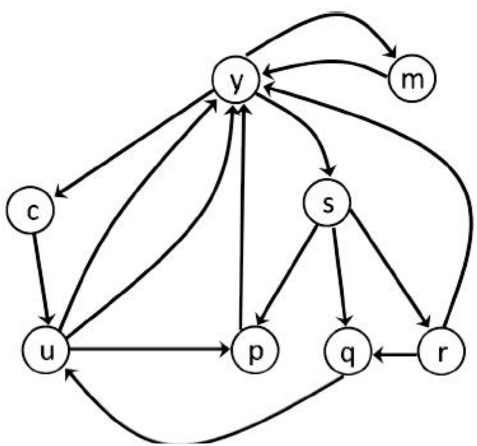
- Took the idea and founded Google, making billions

Jon Kleinberg (professor at Cornell)

- MacArthur Genius Prize, Nevanlina Prize and many other academic honors.

PageRank

- Key idea is hyperlink analysis: take into account the directed graph structure of the web

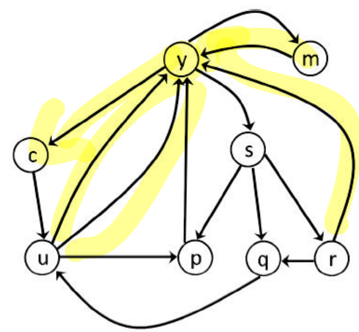


Pagerank

Google
PageRank

Idea 1: “Citations”

As with academic publishing, it’s a good idea to think of each link to a page as a “citation” or “vote of quality”.



Rank pages by in-degree?

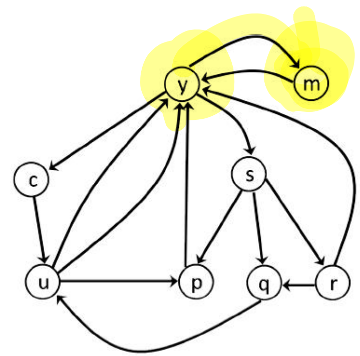
Problems with ranking pages by indegree

- Spamming
- Some linkers not discriminating
- Not all links created equal.

Perhaps we should weight the links somehow and then use the weights of the in-links to rank pages.

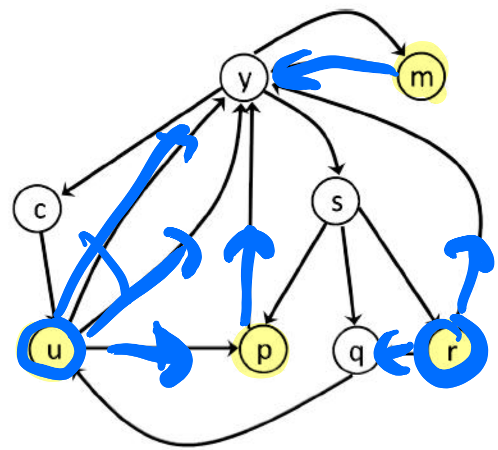
Inching towards Pagerank

- Web page has **high quality** if it’s linked to by lots of **high quality** pages.
- A page is high quality if it links to lots of **high quality** pages.
- So kind of a recursive definition



Inching towards Pagerank

- If web page x has d outgoing links, one of which goes to y, then this contributes 1/d to the importance of y.
- But, we want to take importance/quality of x into account.
- Recursive definition.



q_y quality/importance of web page y.

$$q_y = q_u \frac{2}{3} + q_p 1 + q_r \frac{1}{2} + q_m 1$$

$$P_{ij} = \begin{cases} \frac{1}{d_i} & \text{if } (i \text{ points to } j) \\ 0 & \text{o.w.} \end{cases}$$