Markov Chains and PageRank

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Suppose $\hat{\theta}(X_1, \ldots, X_n)$ is estimator for param θ

MLE estimators for μ, σ^2 of normal

$$\boxed{\hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

for μ unbiased

$$\boxed{\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \hat{\theta}_1)^2}$$

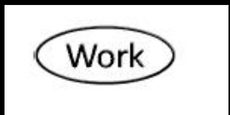
for σ^2 biased.

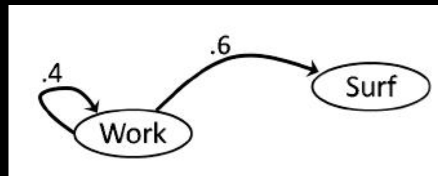
$\hat{\theta}(X_1, \ldots X_n)$ <u>unbiased</u> if

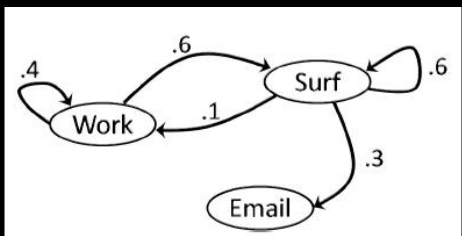
$$E(\hat{\theta}(X_1, \ldots, X_n)) = \theta$$

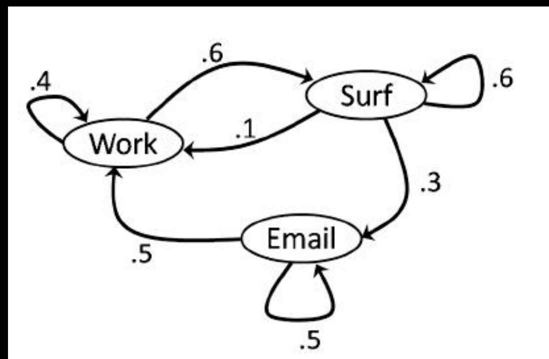
(true param value)

A first Markov chain









My daily life in a nutshell!

Finite Markov Chain

- set of states $S = \{1, 2, \dots, n\}$
- X_t: state at time t
- transition probabilities

$$p_{ij} = \Pr(X_{t+1} = j \mid X_t = i)$$

$S = \{W, S, E\}$

$$\begin{array}{c|ccc} & W & S & E \\ \hline W & & & \\ S & & & \\ E & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} W & S & E \end{array} \\ \begin{array}{c} W \\ S \\ E \end{array} & \begin{pmatrix} .4 & .6 & 0 \\ .1 & .6 & .3 \\ .5 & 0 & .5 \end{pmatrix} \end{array}$$

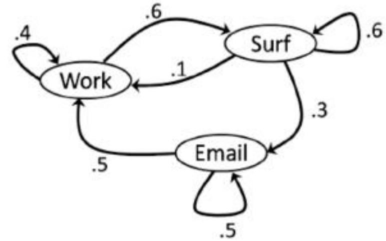
$p_{SE} = 0.3$

Collect these transition probs into n by n matrix P where (i,j) entry of $P = p_{ij}$

sum of entries in ith row: $\sum_{j=1}^{n} p_{ij} = 1 \quad \forall i$

$$\begin{array}{c|ccc} & W & S & E \\ \hline W & .4 & .6 & 0 \\ S & .1 & .6 & .3 \\ E & .5 & 0 & .5 \end{array}$$

$p_{SE} = 0.3$



$q_w^t = Pr(X_t = w \mid X_0 = w)$

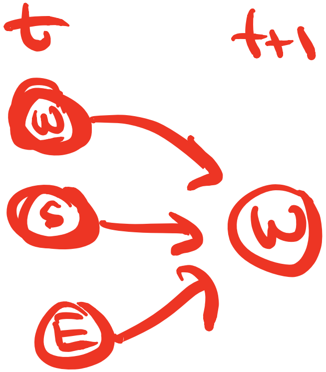
$q_s^t = Pr(X_t = s \mid X_0 = w)$

$q_E^t = Pr(X_t = E \mid X_0 = w)$

t	q_w^t	q_s^t	q_E^t
0	1	0	0
1	0.4	0.6	0
2			

$\boxed{0.4 \cdot 0.4 + 0.6 \cdot 0.1 + 0 \cdot 0.5}$

$0.4 \cdot 0.6 + 0.6 \cdot 0.6 + 0 \cdot 0$



$$\overset{0{,}4 \quad 0{,}6 \quad 0}{\left(q_w^t, q_s^t, q_E^t\right)} \begin{array}{c} W \\ S \\ E \end{array} \begin{pmatrix} .4 & .6 & 0 \\ .1 & .6 & .3 \\ .5 & 0 & .5 \end{pmatrix} = \left(q_w^{t+1}, q_s^{t+1}, q_E^{t+1}\right)$$

$$\boxed{q_w^{t+1} = q_w^t P_{ww} + q_s^t P_{sw} + q_E^t P_{Ew}}$$

$S = \{1, 2, \ldots, n\}$ $\quad P \quad$ TPM $\quad P_{ij} = Pr(X_{t+1} = j \mid X_t = i)$

$$\boxed{q^t = (q_1^t, \ldots, q_n^t)} \qquad \boxed{q_i^t = Pr(X_t = i \mid X_0 = 1)}$$

$$q^0 = (1, 0, 0, \ldots, 0)$$

$$\boxed{q^{t+1} = q^t P}$$

$$\equiv \quad q_i^{t+1} = \sum_{j=1}^{n} q_j^t P_{ji} \qquad \forall i$$

$$LTP = \sum_{j=1}^{n} \Pr(X_t=j \mid X_0=i)\,\Pr(X_{t+1}=i \mid X_t=j)$$

$$q^{(1)} = q^{(0)} P$$

$$q^{(2)} = q^{(1)} P = q^{(0)} P \cdot P = q^{(0)} P^2$$

$$q^{(3)} = q^{(2)} P = q^{(0)} P^2 P = q^{(0)} P^3$$

$$\boxed{q^{(t)} = \underset{(1,0,0,\ldots,0)}{q^{(0)}} P^t}$$

$$(q_1^t, q_2^t, \ldots, q_n^t) = (1,0,0,\ldots,0)\begin{pmatrix} & & \\ & p_{ij} & \\ & & \end{pmatrix}^t$$

$$P^2 = \begin{pmatrix} & & \\ p_{i1} & p_{i2} \cdots & p_{in} \\ & & \end{pmatrix} \begin{pmatrix} & p_{1j} & \\ & p_{2j} & \\ & \vdots & \\ & p_{nj} & \end{pmatrix}$$

$$\underline{P^2_{ij}} = \sum_{k=1}^{n} \underline{p_{ik} p_{kj}} = \sum_{k=1}^{n} P(X_{t+2}=j \mid X_{t+1}=k) \cdot \Pr(X_{t+1}=k \mid X_t=i)$$

$$= \Pr(X_{t+2}=j \mid X_t=i)$$

$$i \to k \to j$$

$$\boxed{P^t_{ij}} = \Pr(X_t=j \mid X_0=i)$$

$$= \Pr(X_{t+c}=j \mid X_c=i) \quad \forall c$$

Prob that in t steps starting at i I end up at j

$$P^2 = \begin{array}{c} \\ W \\ S \\ E \end{array} \begin{array}{c} \begin{array}{ccc} W & S & E \end{array} \\ \begin{pmatrix} .22 & .6 & .18 \\ .25 & .42 & .33 \\ .45 & .3 & .25 \end{pmatrix} \end{array}$$

$$P^3 = \begin{array}{c} \\ W \\ S \\ E \end{array} \begin{array}{c} \begin{array}{ccc} W & S & E \end{array} \\ \begin{pmatrix} .238 & .492 & .270 \\ .307 & .402 & .291 \\ .335 & .450 & .215 \end{pmatrix} \end{array}$$

$$P^{10} \approx \begin{array}{c} \\ W \\ S \\ E \end{array} \begin{array}{c} \begin{array}{ccc} W & S & E \end{array} \\ \begin{pmatrix} .2940 & .4413 & .2648 \\ .2942 & .4411 & .2648 \\ .2942 & .4413 & .2648 \end{pmatrix} \end{array}$$

$Pr(X_{10} = E \mid X_0 = S)$

$$P^{30} \approx \begin{array}{c} \\ W \\ S \\ E \end{array} \begin{array}{c} \begin{array}{ccc} W & S & E \end{array} \\ \begin{pmatrix} .29411764705 & .44117647059 & .26470588235 \\ .29411764706 & .44117647058 & .26470588235 \\ .29411764706 & .44117647059 & .26470588235 \end{pmatrix} \end{array}$$

$$\boxed{P^{60}} \approx \begin{array}{c} \\ W \\ S \\ E \end{array} \begin{array}{c} \begin{array}{ccc} W & S & E \end{array} \\ \begin{pmatrix} .294117647058823 & .441176470588235 & .264705882352941 \\ .294117647068823 & .441176470588235 & .264705882352941 \\ .294117647068823 & .441176470588235 & .264705882352941 \end{pmatrix} \end{array}$$

$Pr(X_{60} = E \mid X_0 = S)$

$Pr(X_{60} = E \mid X_0 = W)$

$Pr(X_{60} = E \mid X_0 = E)$

converging to fixed #'s in each column

$$q^t = q^0 P^t$$

$$\big(\Pr(X_t = 1), \ldots\big) \qquad \begin{pmatrix} 1, 0, 0 \\ 0, 1, 0 \\ \vdots \end{pmatrix}$$

$$\boxed{q^{t+1}} = \boxed{q^t} P$$

$$q^{t+1} = q^t = \pi = (\pi_1, \ldots, \pi_n)$$

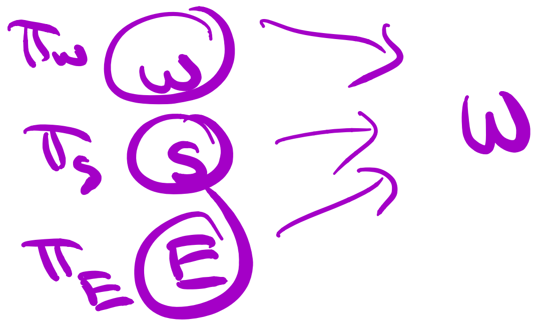
once we reach fixed distribution over states.

$$\boxed{\pi = \pi P}$$

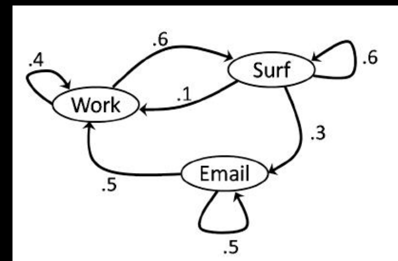
$$\pi_W = \pi_W P_{WW} + \pi_S P_{SW} + \pi_E P_{EW}$$

$$\pi_S = \pi_W P_{WS} + \pi_S P_{SS} + \pi_E P_{ES}$$

$$\pi_E = \pi_W P_{WE} + \pi_S P_{SE} + \pi_E P_{EE}$$



$$\pi[S] = \frac{15}{34}, \pi[W] = \frac{10}{34}, \pi[E] = \frac{9}{34}.$$



$$\begin{cases} \pi[W] & = & .4\pi[W] + .1\pi[S] + .5\pi[E] \\ \pi[S] & = & .6\pi[W] + .6\pi[S] + \ 0\pi[E] \\ \pi[E] & = & 0\pi[W] + .3\pi[S] + .5\pi[E] \end{cases}$$

$$\pi[W] + \pi[S] + \pi[E] = 1$$

Fundamental Thm of MCs.

$$\boxed{\lim_{t\to\infty} P_{ij}^{t} = \pi_j} \quad \forall i$$

P_{ij}^{t} = $Pr(X_t = j \mid X_0 = i)$

π_j: stationary distn

(fine print)

Assume $q_0 = (1, 0, \dots 0)$

$$P_{ij}^{t} = Pr(X_t = j \mid X_0 = i)$$

$$q_i^t = Pr(X_t = i \mid X_0 = 1)$$

$$q^{t+1} = q^t P$$
$$\vdots$$
$$= q^0 P^t$$