5.3 Law of Total Expectation
AGENDA

- Conditional Expectation
- Law of Total Expectation (LTE)
- Law of Total Probability (Continuous version)
Conditional Expectation

**Conditional Expectation:** Let $X$ be a discrete random variable. Then, the conditional expectation of $X$ given $A$ is

$$
E[X \mid A] = \sum_{x \in \Omega_X} x \mathbb{P}(X = x \mid A)
$$

Linearity of expectation still applies to conditional expectation: $E[X + Y \mid A] = E[X \mid A] + E[Y \mid A]$
Law of Total Expectation

Law of Total Expectation (Event Version): Let $X$ be a random variable, and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \mathbb{P}(A_i)$$
LINEARITY OF EXPECTATION APPLIES

To conditional expectation too!!

\[ E(X + Y \mid A) = E(X \mid A) + E(Y \mid A) \]

\[ E(aX + b \mid A) = a \cdot E(X \mid A) + b \]
**Law of total Expectation (RV version)**

**Law of Total Expectation (Event Version):** Let $X$ be a random variable, and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$
\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \mathbb{P}(A_i)
$$

**Law of Total Expectation (RV Version):** Suppose $X$ and $Y$ be discrete random variables. Then,

$$
\mathbb{E}[X] = \sum_{y} \mathbb{E}[X \mid Y = y] p_Y(y)
$$
**Problem**

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are $N$ floors above the ground floor, and if each person is equally likely to get off at any one of the $N$ floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.
Solution

$X$ number of people who enter
$Y$ number of stops

$E(Y) = \sum_{k=0}^{\infty} E(Y|X = k)P(X = k)$

$E(Y|X = k) = E(Y_1 + \ldots + Y_N|X = k)$

$Y_i$ indicates a stop on floor $i$

$E(Y_i|X = k) = (1 - (1 - 1/N)^k)$

$Pr(X = k) = e^{-10} \frac{10^k}{k!}$
Law of total probability (Cont version)
# Multivariate: From Discrete to Continuous

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<thead>
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| **Joint PMF/PDF**        | $p_{X,Y}(x, y) = P(X = x, Y = y)$                                        | $f_{X,Y}(x, y) 
eq P(X = x, Y = y)$                                        |
| **Joint CDF**            | $F_{X,Y}(x, y) = \sum_{t \leq x} \sum_{s \leq y} p_{X,Y}(t, s)$           | $F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t, s) \, ds \, dt$ |
| **Normalization**        | $\sum_{x} \sum_{y} p_{X,Y}(x, y) = 1$                                    | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = 1$ |
| **Marginal PMF/PDF**     | $p_X(x) = \sum_{y} p_{X,Y}(x, y)$                                        | $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy$                      |
| **Expectation**          | $E[g(X, Y)] = \sum_{x} \sum_{y} g(x, y) p_{X,Y}(x, y)$                    | $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) \, dx \, dy$ |
| **Conditional PMF/PDF**  | $p_{X \mid Y}(x \mid y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$                  | $f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$                     |
| **Conditional Expectation** | $E[X \mid Y = y] = \sum_{x} x p_{X \mid Y}(x \mid y)$                    | $E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y) \, dx$  |
| **Independence**         | $\forall x, y, p_{X,Y}(x, y) = p_X(x)p_Y(y)$                            | $\forall x, y, f_{X,Y}(x, y) = f_X(x)f_Y(y)$                               |
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LAW OF TOTAL EXPECTATION (EXAMPLE FROM LAST TIME)

Show that if $X \sim \text{Geo}(p)$, then $\mu = E[X] = 1/p$ by using the LTE conditioning on the first flip.

$$\mu = E[X] = E[X \mid H]P(H) + E[X \mid T]P(T) \quad \text{(LTE)}$$
**Law of Total Expectation (Example from Last Time)**

Show that if $X \sim Geo(p)$, then $\mu = E[X] = 1/p$ by using the LTE conditioning on the first flip.

\[
\mu = E[X] = E[X \mid H]P(H) + E[X \mid T]P(T) \quad \text{(LTE)}
\]

\[
= 1 \cdot p + (E[1 + X]) \cdot (1 - p)
\]

\[
= p + (1 + E[X]) \cdot (1 - p)
\]

So,

\[
\mu = p + (1 + \mu)(1 - p) = p + 1 - p + \mu - \mu p = 1 + \mu - \mu p
\]

\[
0 = 1 - \mu p \rightarrow \mu = \frac{1}{p}
\]