5.3 Law of Total Expectation

Agenda

- Conditional Expectation
- Law of Total Expectation (LTE)
- Law of Total Probability (Continuous version)

Conditional Expectation

Conditional Expectation: Let X be a discrete random variable. Then, the conditional expectation of X given A is

$$\mathbb{E}[X \mid A] = \sum_{x \in \Omega_X} x\mathbb{P}(X = x|A)$$

Linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A]$

Law of Total Expectation

Law of Total Expectation (Event Version): Let X be a random variable, and let events $A_1, ..., A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \, \mathbb{P}(A_i)$$

Linearity of expectation applies

To conditional expectation too!!

$E(X + Y \mid A) = E(X \mid A) + E(Y \mid A)$

$E(aX + b \mid A) = a\,E(X \mid A) + b$

LAW OF TOTAL EXPECTATION (RV VERSION)

Law of Total Expectation (Event Version): Let X be a random variable, and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i]\, \mathbb{P}(A_i)$$

Law of Total Expectation (RV Version): Suppose X and Y be discrete random variables. Then,

$$\mathbb{E}[X] = \sum_{y} \mathbb{E}[X \mid Y = y]\, p_Y(y)$$

Problem

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.

Solution

X number of people who enter

Y number of stops

$$Pr(X = k) = e^{-10} \frac{10^k}{k!}$$

$$E(Y) = \sum_{k=0}^{\infty} E(Y|X = k)P(X = k)$$

$$E(Y|X = k) = E(Y_1 + \ldots + Y_N|X = k)$$

Y_i indicates a stop on floor i

$$E(Y_i|X = k) = \left(1 - (1 - 1/N)^k\right)$$

Law of total probability (Cont version)

Multivariate: From Discrete to Continuous

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = P(X = x, Y = y)$	$f_{X,Y}(x,y) \neq P(X = x, Y = y)$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x} \sum_{s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s) ds dt$
Normalization	$\sum_{x} \sum_{y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_{y} p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Conditional PMF/PDF	$p_{X \mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y = y] = \sum_{x} x p_{X \mid Y}(x \mid y)$	$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y) dx$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$

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Law of Total Expectation (Example from last time)

Show that if $X \sim Geo(p)$, then $\mu = E[X] = 1/p$ by using the LTE conditioning on the first flip.

$$\mu = E[X] = E[X \mid H]P(H) + E[X \mid T]P(T) \qquad \text{(LTE)}$$

Law of Total Expectation (Example from last time)

Show that if $X \sim Geo(p)$, then $\mu = E[X] = 1/p$ by using the LTE conditioning on the first flip.

$$\mu = E[X] = E[X \mid H]P(H) + E[X \mid T]P(T)$$ (LTE)

$$= 1 \cdot p + (E[1 + X]) \cdot (1 - p)$$

$$= p + (1 + E[X]) \cdot (1 - p)$$

So,

$$\cancel{\mu} = p + (1 + \mu)(1 - p) = p + 1 - p + \mu - \mu p = 1 + \cancel{\mu} - \mu p$$

$$0 = 1 - \mu p \rightarrow \mu = \frac{1}{p}$$